ON THE STABILITY OF TYPE I BLOW UP FOR THE ENERGY SUPER CRITICAL HEAT EQUATION

Charles Collot, Pierre Raphaël, Jeremie Szeftel

▶ To cite this version:

Charles Collot, Pierre Raphaël, Jeremie Szeftel. ON THE STABILITY OF TYPE I BLOW UP FOR THE ENERGY SUPER CRITICAL HEAT EQUATION. 2016. hal-01324005

HAL Id: hal-01324005
https://hal.science/hal-01324005

Preprint submitted on 31 May 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1605.07337v1 [math.AP] 24 May 2016

ON THE STABILITY OF TYPE I BLOW UP FOR THE ENERGY SUPER CRITICAL HEAT EQUATION

CHARLES COLLOT, PIERRE RAPHAËL, AND JEREMIE SZEFTEL

Abstract. We consider the energy super critical semilinear heat equation

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^3, \quad p > 5.$$

We first revisit the construction of radially symmetric self similar solutions performed through an ode approach in [51], [2], and propose a bifurcation type argument suggested in [3] which allows for a sharp control of the spectrum of the corresponding linearized operator in suitable weighted spaces. We then show how the sole knowledge of this spectral gap in weighted spaces implies the finite codimensional non radial stability of these solutions for smooth well localized initial data using energy bounds. The whole scheme draws a route map for the derivation of the existence and stability of self similar blow up in non radial energy super critical settings.

1. Introduction

1.1. **Setting of the problem.** We consider the focusing nonlinear heat equation

$$\left\{\begin{array}{l} \partial_t u = \Delta u + |u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u_{|t=0} = u_0, \end{array}\right. \tag{1.1}$$

where $p > 1$. This model dissipates the total energy

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int u^{p+1}, \quad \frac{1}{2}\frac{dE}{dt} = -\int (\partial_t u)^2 < 0 \tag{1.2}$$

and admits a scaling invariance: if $u(t,x)$ is a solution, then so is

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.3}$$

This transformation is an isometry on the homogeneous Sobolev space

$$\|u_\lambda(t,\cdot)\|_{\dot H^{s_c}} = \|u(t,\cdot)\|_{\dot H^{s_c}} \text{ for } s_c = \frac{d}{2} - \frac{2}{p-1}.$$

We address in this paper the question of the existence and stability of blow up dynamics in the energy super critical range $s_c > 1$ emerging from well localized initial data.

1.2. **Type I and type II blow up.** There is a large litterature devoted to the question of the description of blow up solutions for (1.1) and we recall some key facts related to our analysis.

Type I blow-up. The universal scaling lower bound on blow up rate

$$\|u(t,\cdot)\|_{L^\infty} \gtrsim \frac{1}{(T-t)^{\frac{1}{p-1}}}$$

2010 *Mathematics Subject Classification.* primary, 35K58 35B44 35B35, secondary 35J61 35B32.
Key words and phrases. blow-up, existence, heat, self-similar, stability, supercritical.

is saturated by Type I singularities:

$$\|u(t,\cdot)\|_{L^\infty}\sim\frac{1}{(T-t)^{\frac{1}{p-1}}}.$$

These solutions concentrate to leading order a blow up profile

$$u(t,x)\sim\frac{1}{\lambda(t)^{\frac{2}{p-1}}}v\left(\frac{x}{\lambda(t)}\right),\quad \lambda(t)=\sqrt{T-t},$$

which solves the non linear elliptic equation

$$\Delta v-\frac{1}{2}\Lambda v+v^p=0,\quad \Lambda v=\frac{2}{p-1}v+y\cdot\nabla v. \tag{1.4}$$

There are three known classes of radial solutions to (1.4):

- the constant solution
$$\kappa=\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$
which generates the stable ODE type blow up [20, 21, 22, 23, 39, 40];
- the singular at the origin homogeneous self similar solution
$$\Phi_*=\frac{c_\infty}{|x|^{\frac{2}{p-1}}},\quad c_\infty=\left(\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right)^{\frac{1}{p-1}}; \tag{1.5}$$
- for
$$1+\frac{4}{d-2}<p<p_{JL}=\begin{cases}+\infty & \text{for } d\leq 10,\\ 1+\frac{4}{d-4-2\sqrt{d-1}} & \text{for } d\geq 11,\end{cases} \tag{1.6}$$
where p_{JL} is the so called Joseph-Lundgren exponent, there exists a quantized sequence of smooth radially symmetric solutions Φ_n to (1.4) which behave like
$$\Phi_n(r)\sim\frac{c_n}{r^{\frac{2}{p-1}}}\quad\text{as}\quad r\to+\infty.$$
These solutions have been constructed using global Lyapounov functionals based ODE methods, [30, 51, 1, 2], and a sharp condition for their existence in the radial positive class is given in [44].

Note that all these profiles have infinite energy and it is not clear how they may participate in singularity formation emerging from smooth well localized initial data. In the radially symmetric setting, the series of breakthrough works [35, 36] gives partial answers showing the universality of the ODE blow up, and the possiblity of threshold dynamics with Φ^* or Φ_n regimes depending on the value of p. The analysis however is strongly restricted to the radial setting and uses the intersection number Lyapounov functionals based on the maximum principle. In particular this approach does not provide any insight into the direct construction of these blow up profiles and their dynamical stability in the non radial setting.

Type II blow-up. For $p>p_{JL}$, there exist type II blow-up solutions

$$\lim_{t\to T}\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}}=+\infty.$$

They appear in the radial setting as threshold dynamics again at the boundary of the ODE blow up set, [37], and dynamical proofs were proposed in [25, 43, 45]. Their construction has been revisited in [42, 6] in the setting of dispersive Schrödinger and wave equations, and in [7] for the non radial heat equation, to produce the full quantized sequence of smooth type II blow up bubbles. The blow-up profile near the singularity is a stationary profile:

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) \ll \sqrt{T-t}$$

where Q solves the soliton equation:

$$\Delta Q + Q^p = 0. \tag{1.7}$$

The heart of the analysis is to control the flow near Q using suitable energy estimates, hence avoiding maximum principle tools or spectral arguments. Type II is intimately connected to the singular self similar profile (1.5), see [25, 42] for a discussion on this fundamental matter.

1.3. **Statement of the result.** Our aim in this paper is to propose a robust approach for both the existence and stability of self similar blow up with smooth self similar Φ_n like profile. For the sake of simplicity, we restrict ourselves to

$$d = 3, \quad p > 5, \quad p_{JL} = +\infty. \tag{1.8}$$

We first revisit the construction of self similar blow up solutions of [51, 2] and implement an abstract bifurcation argument which relies on the sole existence of the stationary profile Q given by (1.7). Note that this kind of argument is classical in the ODE literature, see for example [1, 11, 9], and relies on the oscillatory nature of the eigenfunctions of the linearized operator close to Φ^* for $p < p_{JL}$.

Proposition 1.1 (Existence and asymptotic of excited self similar solutions)**.** *Assume* (1.8)*. For all $n > N$ large enough, there exist a*[1] *smooth radially symmetric solution to the self similar equation* (1.4) *such that*

$$\Lambda\Phi_n \quad \textit{vanishes exactly } n \textit{ times on} \quad (0, +\infty).$$

Moreover, there exists a small enough constant $r_0 > 0$ independent of n such that:
1. Behavior at infinity*:*

$$\lim_{n\to+\infty} \sup_{r\geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0. \tag{1.9}$$

2. Behaviour at the origin*: there exists a sequence $\mu_n > 0$ with $\mu_n \to 0$ as $n \to +\infty$ such that*

$$\lim_{n\to+\infty} \sup_{r\leq r_0} \left| \Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right) \right| = 0. \tag{1.10}$$

Hence these solutions realize a connection between the ground state behavior Q at the origin, and the homogeneous self similar decay Φ_* at infinity. We now claim that these solutions are the blow up profile of a class of finite energy initial data leaving on a non radial n codimensional manifold.

Theorem 1.2 (Finite codimensional stability of Φ_n)**.** *Assume* (1.8)*. Let $n > N$ large enough. There exists a Lipschitz codimension n manifold*[2] *of non radial initial data with finite energy*

$$u_0 = \chi_{A_0}\Phi_n + w_0$$

where $A_0 \gg 1$ is large enough and w_0 is small enough[3]

$$\|w_0\|_{H^2_\rho} + \|\Delta w_0\|_{L^2} + \|w_0\|_{\dot{H}^{s_c}} \ll 1, \tag{1.11}$$

[1] locally unique in some suitable space

[2] see Proposition 4.9 for a precise statement of the Lipschitz regularity.

[3] See (1.18) and below for the definition of the weighted Sobolev space H^2_ρ.

such that the corresponding solution to (1.1) *blows up in finite time* $0 < T < +\infty$ *with a decomposition*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(\Phi_n + v)\left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

where:
1. Control of the geometrical parameters*: the blow up speed is self similar*

$$\lambda(t) = \sqrt{(2 + o(1))(T - t)} \quad as \quad t \to T$$

and the blow up point converges

$$x(t) \to x(T) \quad as \quad t \to T. \tag{1.12}$$

2. Behaviour of Sobolev norms*: there holds the asymptotic stability of the self similar profile above scaling*

$$\lim_{t \to T} \|v(t)\|_{\dot{H}^s} = 0 \quad for \quad s_c < s \le 2, \tag{1.13}$$

the boundedness of norms below scaling

$$\limsup_{t \to T} \|u(t)\|_{\dot{H}^s} < +\infty \quad for \quad 1 \le s < s_c, \tag{1.14}$$

and the logarithmic growth of the critical norm

$$\|u(t)\|_{\dot{H}^{s_c}} = c_n(1 + o_{t \to T}(1))\sqrt{|\log(T - t)|}, \quad c_n \neq 0. \tag{1.15}$$

Comments on the results.

1. On the construction of self similar solutions. The construction of self similar solutions has been performed in [51, 2] using a global Lyapounov functional ode approach. A very interesting variational approach has also been developed in [5, 19] in the setting of the related wave map problem. But there are many classical problems which lack both the variational structure and the monotonicity formulas, hence the need for a more systematic approach typically connected in a way or another to a bifurcation argument, which is the method we are implementing here. This procedure has been applied in various settings, see for example [1, 11]. One advantage is that the proof further allows for a control of the linearized operator near the bifurcated object. The prize to pay however is that we only get the bifurcated family locally near the bifurcation point, and not the whole branch[4], in particular not the fundamental mode. A closely related theorem is the construction [29] for the KdV equation near the critical exponent.

2. Stability of self similar blow up. There is an important literature devoted to the stability of self similar solutions for both parabolic and dispersive problems. We aim at developing a robust approach which will extend to more complicated systems. Hence we avoid on purpose maximum principle like tools. In [13, 14, 15, 16], this kind of question has also been addressed for the radially symmetric supercritical wave map problem, Yang-Mills, wave equation and Yang-Mills heat flow. In those works, the analysis requires a detailed description of the complex spectrum of the linearized operator in suitable spaces which is a delicate matter, and seems to rely heavily on the fact that in the cases under consideration, the self similar solution has an explicit formula. Our approach is different: once we know the spectral gap estimate with exponential weight which is an elementary consequence of either the

[4]unless one works for $p_{JL} - \varepsilon < p < p_{JL}$ in which case the whole family could be bifurcated along the same lines as for the supercritical gKdV equation performed in [29].

variational characterization of the self similar solution as in [5], or the construction of the solution by bifurcation as in the setting of Proposition 1.1, then the control of the nonlinear flow follows by adapting the general strategy based on energy bounds of [48, 42]. In fact, the exponential decay bounds behind (1.13) considerably simplify the analysis with respect to the study of type II blow up. The connexion with type II blow up has been made in [26] using exponential weights again, and the analysis is indeed intrinsically more involved. This energy method in weighted spaces also draws a natural connexion with the analysis of ODE type I blow up for both the heat and the wave equation [21, 39, 41]. Note also that we assume (1.8) for the sake of simplicity only[5]. The solutions of Theorem 1.2 will be obtained using first a by now classical Brouwer like topological argument [10, 42], which is then complemented by a local uniqueness statement to construct the Lipschitz manifold as in [6, 33, 28, 17].

3. The flow near the ground state. The question of the classification of the flow near the special class of stationary solutions Q has attracted a considerable attention in the past ten years in connection with the construction of the unstable manifold [46], or the complete classification of the flow near Q in energy subcritical [46, 34] and critical settings [8]. The corresponding instabilities are central in the derivation of unstable type II blow up bubbles, [42]. From (1.10), the self similar solution ressembles the solitary wave Q up to scaling near the origin, and hence the stability Theorem 1.2 can be viewed as describing one instability of the solitary wave solution in a suitable function space. Here a fundamental issue is that the linearized operator $H = -\Delta - pQ^{p-1}$ is *unbounded from below* in the sense of quadratic forms for $p < p_{JL}$. This is a major difference with respect to the case $p > p_{JL}$ where $H > 0$. Our analysis in this paper shows how the nonlinear bifurcated solution Φ_n precisely allows for the suitable modification of the linearized operator which fixes this unboundedness from below of H. One also observes the same behaviour of Sobolev norms (1.13), (1.14) as in [42] which illustrates the deep non trivial structure in space of the associated blow up scenario[6]. Let us also stress that the nature of our energy like non linear estimates goes far beyond the stability issues of specific dynamics, and has allowed in [34] in a dispersive setting and [8] in the parabolic setting for a complete description of the flow near the ground states.

This paper and [48, 42, 6, 7] hence display a deep unity and design a route map based on robust energy estimates for the proof of the existence and stability of type I or type II blow up bubbles in both radial and non radial settings.

Acknowledgements. All three authors are supported by the ERC-2014-CoG 646650 SingWave. P.R. is a junior member of the Institut Universitaire de France.

Notations. From now on and for the rest of this paper we fix

$$d = 3, \quad p > 5.$$

The ground state expansion. We let Φ_* given by (1.5) be the unique radial homogenous self similar solution to (1.4). We let $Q(r)$ denote the unique radially symmetric solution to

$$\left\{ \begin{array}{l} Q'' + \frac{2}{r}Q' + Q^p = 0, \\ Q(0) = 1, \quad Q'(0) = 0, \end{array} \right.$$

[5] Raising dimensions causes the nonlinearity to become non smooth since $\lim_{d\to+\infty} p_{JL} = 1$ and hence would lead to additional but manageable technical difficulties.

[6] and hence its relevance in particular for more geometric problems like the harmonic heat flow of surfaces.

which asymptotic behavior at infinity is from standard ODE argument[7] given by

$$Q(r) = (1 + o_{r\to+\infty}(1))\Phi_*(r).$$

The next term is this expansion relates to the p_{JL} exponent (1.6) which is infinite in dimension $d = 3$. Hence the quadratic polynomial

$$\gamma^2 - \gamma + pc_\infty^{p-1} = 0$$

has complex roots

$$\gamma = \frac{1}{2} \pm i\omega, \quad \Delta := 1 - 4pc_\infty^{p-1} < 0, \quad \omega := \frac{\sqrt{-\Delta}}{2} \tag{1.16}$$

and the asymptotic behavior of Q may be precised[8]:

$$Q(r) = \Phi_*(r) + \frac{c_1 \sin(\omega \log(r) + c_2)}{r^{\frac{1}{2}}} + o\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty \tag{1.17}$$

where $c_1 \neq 0$ and $c_2 \in \mathbb{R}$. Note that

$$\frac{1}{2} - \frac{2}{p-1} = s_c - 1 > 0$$

so that the second term in the expansion of Q is indeed a correction term.

Weighted spaces. We define the derivation operator

$$D^k := \begin{cases} \Delta^m & \text{for } m = 2k, \\ \nabla\Delta^k & \text{for } m = 2k+1. \end{cases}$$

We define the scalar product

$$(f, g)_\rho = \int_{\mathbb{R}^3} f(x)g(x)\rho dx, \quad \rho = e^{-\frac{|x|^2}{2}} \tag{1.18}$$

and let L^2_ρ be the corresponded weighted L^2 space. We let H^k_ρ be the completion of $\mathcal{C}^\infty_c(\mathbb{R}^d)$ for the norm

$$\|u\|_{H^k_\rho} = \sqrt{\sum_{j=0}^{k} \|D^j u\|^2_{L^2_\rho}}.$$

Linearized operators. The scaling semi-group on functions $u : \mathbb{R}^d \to \mathbb{R}$:

$$u_\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x) \tag{1.19}$$

has for infinitesimal generator the linear operator

$$\Lambda u := \frac{2}{p-1}u + x.\nabla u = \frac{\partial}{\partial\lambda}(u_\lambda)_{|\lambda=1}.$$

We define the linearized operator corresponding to (1.4) around respectively Φ_* and Φ_n by

$$\mathcal{L}_\infty := -\Delta + \Lambda - \frac{pc_\infty^{p-1}}{r^2}, \quad \mathcal{L}_n := -\Delta + \Lambda - p\Phi_n^{p-1}$$

and their projection onto spherical harmonics:

$$\mathcal{L}_{\infty,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_*^{p-1}, \quad m \in \mathbb{N},$$

$$\mathcal{L}_{n,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_n^{p-1}, \quad m \in \mathbb{N}.$$

[7] see [12, 27, 31].

[8] see [12, 27, 31].

Note that $\mathcal{L}_\infty$ is formally self adjoint for the L^2_ρ scalar product but (1.16) implies that the associated quadratic form is not bounded from below[9] on H^1_ρ. We similarly define the linearized operator corresponding to (1.7) around Q:

$$\begin{aligned} H &:= -\Delta - pQ^{p-1} \\ H_m &:= -\partial_{rr} - \frac{2}{r}\partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}, \quad m \in \mathbb{N}. \end{aligned}$$

and again H is not bounded from below on $\dot{H}^1$.

General notation. We let $\chi(x)$ denote a smooth radially symmetric function with

$$\chi(x) := \begin{cases} 1 & \text{for } |x| \le \frac{1}{4}, \\ 0 & \text{for } |x| \ge \frac{1}{2}, \end{cases}$$

and for $A > 0$ (note the difference with (1.19)),

$$\chi_A(x) = \chi\left(\frac{x}{A}\right).$$

Organization of the paper. This paper is organized as follows. In section 2, we construct the family of self similar solutions Φ_n using a nonlinear matching argument. The argument is classical, but requires a careful track of various estimates to obtain the sharp bounds (1.9), (1.10). In section 3, we show how these bounds coupled with Sturm-Liouville like arguments allow for a sharp counting of the number of instabilities of the linearized operator close to Φ_n which is self adjoint against the confining measure $\rho(y)dy$, Proposition 3.1. In section 4, we turn to the heart of the dynamical argument and show how the spectral estimates in the weighted space coupled with the control of the super critical $\dot{H}^2$ norm design a stability zone for well localized initial data.

2. Construction of self-similar profiles

Our aim in this section is to construct radially symmetric solutions to the self similar equation

$$\Delta v - \Lambda v + v^p = 0, \tag{2.1}$$

by using the classical strategy of gluing solutions which behave like Φ_* at infinity, and like Q at the origin. As in [3, 11, 1], the matching is made possible by the oscillatory behaviour (1.17) for $p < p_{JL}$. The strength of this approach it that it relies on the implicit function theorem and not on fine monotonicity properties, and in this sense it goes far beyond the scalar parabolic setting, see for example [29] for a deeply related approach. The sharp control of the obtained solution (1.9), (1.10) will allow us to control the eigenvalues of the associated linearized operator in suitable exponentially weighted spaces, see Proposition 3.1.

2.1. **Exterior solutions.**

Recall that Φ_* given by (1.5) is a solution to (2.1) on $(0, +\infty)$. Our aim in this section is to construct the full family of solutions to (2.1) on $[r_0, +\infty)$ for some small $r_0 > 0$ with the suitable behaviour at infinity. The argument is a simple application of the implicit function theorem and continuity properties of the resolvent of $\mathcal{L}_\infty$ in suitable weighted spaces.

[9] this is a limit point circle case as $r \to 0$, [49].

Given $0<r_0<1$, we define X_{r_0} as the space of functions on $(r_0,+\infty)$ such that the following norm is finite

$$\|w\|_{X_{r_0}}=\sup_{r_0\leq r\leq 1} r^{\frac12}|w|+\sup_{r\geq 1} r^{\frac{2}{p-1}+2}|w|.$$

Lemma 2.1 (Outer resolvent of $\mathcal{L}_\infty$)**.** 1. Basis of fundamental solutions*: there exists two solutions ψ_1 and ψ_2 of*

$$\mathcal{L}_\infty(\psi_j)=0 \text{ for } j=1,2 \quad \text{on} \quad (0,+\infty) \tag{2.2}$$

with the following asymptotic behavior:

$$\psi_1=\frac{1}{r^{\frac{2}{p-1}}}\left(1+O\left(\frac{1}{r^2}\right)\right),\ \psi_2=r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\left(1+O\left(\frac{1}{r^2}\right)\right),\ \text{as } r\to+\infty \tag{2.3}$$

and

$$\psi_1=\frac{c_3\sin(\omega\log(r)+c_4)}{r^{\frac12}}+O\left(r^{\frac32}\right),\ \psi_2=\frac{c_5\sin(\omega\log(r)+c_6)}{r^{\frac12}}+O\left(r^{\frac32}\right)\ \text{ as } r\to 0 \tag{2.4}$$

where $c_3,c_5\neq 0$ and $c_4,c_6\in\mathbb{R}$. Moreover, there exists $c\neq 0$ such that

$$\Lambda\psi_1=\frac{c}{r^{\frac{2}{p-1}+2}}\left(1+O\left(\frac{1}{r^2}\right)\right)\ \text{ as } r\to+\infty. \tag{2.5}$$

2. Continuity of the resolvent*: let the inverse*

$$\mathcal{T}(f)=\left(\int_r^{+\infty} f\psi_2 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_1-\left(\int_r^{+\infty} f\psi_1 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_2,$$

then

$$\mathcal{L}_\infty(\mathcal{T}(f))=f$$

and

$$\|\mathcal{T}(f)\|_{X_{r_0}}\lesssim\int_{r_0}^1|f|r'^{\frac32}dr'+\sup_{r\geq 1} r^{\frac{2}{p-1}+2}|f|. \tag{2.6}$$

Proof. The proof is classical and we sketch the details for the reader's convenience.

step 1 Basis of homogeneous solutions. Recall (1.16). Let the change of variable and unknown

$$\psi(r)=\frac{1}{y^{\frac{\gamma}{2}}}\phi(y),\quad y=r^2,$$

then

$$\partial_r=2r\partial_y,\quad \partial_r^2=4r\partial_y(r\partial_y)=4r^2\partial_y^2+4r\partial_y(r)\partial_y=4y\partial_y^2+2\partial_y,\quad r\partial_r=2y\partial_y.$$

This yields

$$\mathcal{L}_\infty(\psi)\ =\ \left(-4y\partial_y^2-2\partial_y-4\partial_y+\frac{2}{p-1}+2y\partial_y-\frac{pc_\infty^{p-1}}{y}\right)\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right).$$

Since

$$\begin{aligned}\partial_y\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right)\ &=\ \frac{1}{y^{\frac{\gamma}{2}}}\phi'(y)-\frac{\gamma}{2y^{\frac{\gamma}{2}+1}}\phi(y),\\ \partial_y^2\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right)\ &=\ \frac{1}{y^{\frac{\gamma}{2}}}\phi''(y)-\frac{\gamma}{y^{\frac{\gamma}{2}+1}}\phi'(y)+\frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{y^{\frac{\gamma}{2}+2}}\phi(y),\end{aligned}$$

we infer

$$\begin{aligned}
\mathcal{L}_\infty(\psi) &= \Bigg\{ -4y\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi''(y) - \frac{\gamma}{y^{\frac{\gamma}{2}+1}}\phi'(y) + \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{y^{\frac{\gamma}{2}+2}}\phi(y)\right)\\
&+ (-6+2y)\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi'(y) - \frac{\gamma}{2y^{\frac{\gamma}{2}+1}}\phi(y)\right) + \left(\frac{2}{p-1} - \frac{pc_\infty^{p-1}}{y}\right)\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\Bigg\}\\
&= \frac{1}{y^{\frac{\gamma}{2}}}\Bigg\{ -4y\phi''(y) + \Big(4\gamma - 6 + 2y\Big)\phi'(y)\\
&+ \left(\frac{2}{p-1} - \gamma + \left(3\gamma - \gamma(\gamma+2) - pc_\infty^{p-1}\right)\frac{1}{y}\right)\phi(y)\Bigg\}.
\end{aligned}$$

Since γ satisfies

$$\gamma^2 - \gamma + pc_\infty^{p-1} = 0,$$

we infer

$$\mathcal{L}_\infty(\psi) = -\frac{4}{y^{\frac{\gamma}{2}}}\left\{ y\phi''(y) + \left(-\gamma + \frac{3}{2} - \frac{y}{2}\right)\phi'(y) + \frac{1}{4}\left(-\frac{2}{p-1} + \gamma\right)\phi(y)\right\}.$$

We change again variable by setting

$$\phi(y) = w(z), \quad z = \frac{y}{2}.$$

We have

$$\phi'(y) = \frac{1}{2}w'(z), \quad \phi''(y) = \frac{1}{4}w''(z)$$

and obtain

$$\mathcal{L}_\infty(\psi) = -\frac{2}{y^{\frac{\gamma}{2}}}\left(zw''(z) + \left(-\gamma + \frac{3}{2} - z\right)w'(z) - \left(\frac{1}{p-1} - \frac{\gamma}{2}\right)w(z)\right).$$

Thus, $\mathcal{L}_\infty(\psi) = 0$ if and only if

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0 \tag{2.7}$$

where we have used the notations

$$a = \frac{1}{p-1} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}. \tag{2.8}$$

(2.7) is known as Kummer's equation. As long as a is not a negative integer - which holds in particular for our choice of a in (2.8) -, a basis of solutions to Kummer's equation consists of the Kummer's function $M(a,b,z)$ and the Tricomi function $U(a,b,z)$. These special functions have the following asymptotic behavior for $z \geq 0$ (see for example [47])

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}z^{a-b}e^z(1+O(z^{-1})), \ U(a,b,z) = z^{-a}(1+O(z^{-1})) \text{ as } z \to +\infty, \tag{2.9}$$

$$M(a,b,z) = 1 + O(z) \text{ as } z \to 0, \tag{2.10}$$

and[10] for $1 \leq \Re(b) < 2$ with $b \neq 1$,

$$U(a,b,z) = \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(z^{2-\Re(b)}) \text{ as } z \to 0. \tag{2.11}$$

[10]Note that our choice of b in (2.8) is such that $\Re(b) = 1$ and $b \neq 1$.

Since w is a linear combination of $M(a,b,z)$ and $U(a,b,z)$, we immediately infer from (2.9), (2.10) and (2.11) the asymptotic of w both as $z\to+\infty$ and $z\to 0_+$. Finally, since

$$\psi(r)=\frac{1}{r^\gamma}w\left(\frac{r^2}{2}\right),$$

we infer from the asymptotic of w the claimed asymptotic for ψ both as $r\to+\infty$ and $r\to 0_+$. This concludes the proof of (2.3), (2.4).

step 2 Estimate on the resolvent. The Wronskian

$$W \quad:=\quad \psi_1'\psi_2-\psi_2'\psi_1.$$

satisfies

$$W'=\left(-\frac{2}{r}+r\right)W,\quad W=\frac{C}{r^2}e^{\frac{r^2}{2}}$$

where we may without loss of generality assume $C=1$. We then solve

$$\mathcal{L}_\infty(w)=f$$

using the variation of constants which yields

$$w=\left(a_1+\int_r^{+\infty}f\psi_2 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_1+\left(a_2-\int_r^{+\infty}f\psi_1 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_2. \tag{2.12}$$

In particular, $\mathcal{T}(f)$ corresponds to the choice $a_1=a_2=0$ and thus satisfies

$$\mathcal{L}_\infty(\mathcal{T}(f))=f.$$

Next, we estimate $\mathcal{T}(f)$ using the asymptotic behavior (2.3) and (2.4) of ψ_1 and ψ_2 as $r\to 0_+$ and $r\to+\infty$. For $r\geq 1$, we have

$$\begin{aligned}
& r^{\frac{2}{p-1}+2}|\mathcal{T}(f)|\\
= \; & r^{\frac{2}{p-1}+2}\left|\left(\int_r^{+\infty}f\psi_2 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_1-\left(\int_r^{+\infty}f\psi_1 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_2\right|\\
\lesssim \; & r^2\left(\int_r^{+\infty}|f|r'^{\frac{2}{p-1}-1}dr'\right)+r^{\frac{4}{p-1}-1}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}|f|\frac{1}{r'^{\frac{2}{p-1}}}r'^2e^{-\frac{r'^2}{2}}dr'\right)\\
\lesssim \; & \left\{\sup_{r>1}\left(r^2\left(\int_r^{+\infty}\frac{dr'}{r'^3}\right)+r^{\frac{4}{p-1}-1}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}r'^{-\frac{4}{p-1}}e^{-\frac{r'^2}{2}}dr'\right)\right)\right\}\sup_{r\geq 1}r^{\frac{2}{p-1}+2}|f|\\
\lesssim \; & \sup_{r\geq 1}r^{\frac{2}{p-1}+2}|f|.
\end{aligned}$$

Also, for $r_0\leq r\leq 1$, we have

$$\begin{aligned}
& r^{\frac12}\left|\left(\int_r^{+\infty}f\psi_2 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_1-\left(\int_r^{+\infty}f\psi_1 r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_2\right|\\
\lesssim \; & \int_r^1|f|r'^{\frac32}dr'+\int_1^{+\infty}r'^{\frac{2}{p-1}-1}|f|dr'\lesssim\int_{r_0}^1|f|r'^{\frac32}dr'+\sup_{r\geq 1}r^{\frac{2}{p-1}+2}|f|
\end{aligned}$$

and (2.6) is proved.

step 3 Refined control of ψ_1. We now turn to the proof of (2.5). We decompose

$$\psi_1=\frac{1}{r^{\frac{2}{p-1}}}+\widetilde{\psi}_1. \tag{2.13}$$

Since $\mathcal{L}_\infty(\psi_1)=0$, we infer

$$\mathcal{L}_\infty(\widetilde{\psi}_1)=f$$

where f is given by

$$
\begin{aligned}
f &= -\mathcal{L}_\infty\left(\frac{1}{r^{\frac{2}{p-1}}}\right) = \partial_r^2\left(\frac{1}{r^{\frac{2}{p-1}}}\right) + \frac{2}{r}\partial_r\left(\frac{1}{r^{\frac{2}{p-1}}}\right) + \frac{pc_\infty^{p-1}}{r^2}\frac{1}{r^{\frac{2}{p-1}}}\\
&= \frac{2(p-3)}{p-1)}\frac{1}{r^{\frac{2}{p-1}+2}}.
\end{aligned}
$$

In view of (2.12), we infer

$$
\widetilde{\psi}_1 = \left(a_1 + \frac{2(p-3)}{p-1)}\int_r^{+\infty}\psi_2\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_1 + \left(a_2 - \frac{2(p-3)}{p-1}\int_r^{+\infty}\psi_1\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_2.
$$

On the other hand, we deduce from the asymptotic behavior of ψ_1

$$
\widetilde{\psi}_1 = o\left(\frac{1}{r^{\frac{2}{p-1}}}\right) \text{ as } r \to +\infty.
$$

In view of the asymptotic behavior of ψ_1 and ψ_2 as $r \to +\infty$, this forces $a_1 = a_2 = 0$ and hence

$$
\widetilde{\psi}_1 = \frac{2(p-3)}{p-1}\left(\int_r^{+\infty}\psi_2\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_1 - \frac{2(p-3)}{p-1}\left(\int_r^{+\infty}\psi_1\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_2.
$$

Then, applying Λ to both sides, and using the asymptotic behavior of ψ_1 and ψ_2 as $r \to +\infty$ yields

$$
\Lambda\widetilde{\psi}_1 = \frac{c}{r^{\frac{2}{p-1}+2}}\left(1 + O\left(\frac{1}{r^2}\right)\right) \text{ as } r \to +\infty
$$

for some constant[11] $c \neq 0$. Injecting this into (2.13) yields

$$
\Lambda\psi_1 = \Lambda\widetilde{\psi}_1 = \frac{c}{r^{\frac{2}{p-1}+2}}\left(1 + O\left(\frac{1}{r^2}\right)\right) \text{ as } r \to +\infty
$$

for some constant $c \neq 0$ and concludes the proof of Lemma 2.1. □

We are now in position to construct the family of outer self similar solutions as a classical consequence of the implicit function theorem.

Proposition 2.2 (Exterior solutions). *Let $0 < r_0 < 1$ a small enough universal constant. For all*

$$
0 < \varepsilon \ll r_0^{s_c-1}, \tag{2.14}
$$

there exists a solution u to

$$
\Delta u - \Lambda u + u^p = 0 \quad on \quad (r_0, +\infty) \tag{2.15}
$$

of the form

$$
u = \Phi_* + \varepsilon\psi_1 + \varepsilon w
$$

with the bounds:

$$
\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}, \quad \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}. \tag{2.16}
$$

Furthermore,

$$
w_{|\varepsilon=0} = 0 \ and \ \|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}.
$$

[11] Actually, c is explicitly given by

$$
c = -\frac{p-3}{p-1} \neq 0.
$$

Proof. This a classical consequence of Lemma 2.1.

step 1. Setting up the Banach fixed point. Let v such that

$$u = \Phi_* + \varepsilon v,$$

then u solves (2.15) iff:

$$\mathcal{L}_\infty(v) = \varepsilon \frac{p(p-1)}{2}\Phi_*^{p-2}v^2 + \varepsilon F(\Phi_*, v, \varepsilon) \text{ on } r > r_0,$$

where

$$F(\Phi_*, v, \varepsilon) = \frac{1}{\varepsilon^2}\left((\Phi_* + \varepsilon v)^p - \Phi_*^p - p\Phi_*^{p-1}\varepsilon v - \frac{p(p-1)}{2}\Phi_*^{p-2}\varepsilon^2 v^2\right).$$

Furthermore, we decompose

$$v = \psi_1 + w$$

and hence, using in particular the fact that $\mathcal{L}_\infty(\psi_1) = 0$, w is a solution to

$$\mathcal{L}_\infty(w) = p(p-1)\varepsilon G[\Phi_*, \psi_1, \varepsilon]w \text{ on } r > r_0$$

where we defined the map:

$$G[\Phi_*, \psi_1, \varepsilon]w = \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1 + w))^{p-2}ds\right)(\psi_1 + w)^2.$$

We claim the non linear bounds: assume that

$$\|w\|_{X_{r_0}} \leq 1,$$

then

$$\int_{r_0}^1 |G[\Phi_*, \psi_1, \varepsilon]w| r'^{\frac{3}{2}}dr' + \sup_{r\geq 1} r^{\frac{2}{p-1}+2}|G[\Phi_*, \psi_1, \varepsilon]w| \lesssim r_0^{1-s_c} \tag{2.17}$$

and

$$\begin{aligned} &\int_{r_0}^1 |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| r'^{\frac{3}{2}}dr' \\ &+ \sup_{r\geq 1} r^{\frac{2}{p-1}+2}|G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\ \lesssim\ & r_0^{1-s_c}\|w_1 - w_2\|_{X_{r_0}}. \end{aligned} \tag{2.18}$$

Assume (2.17), (2.18), then we look for w as the solution of the following fixed point

$$w = \varepsilon p(p-1)\mathcal{T}\Big(G[\Phi_*, \psi_1, \varepsilon]w\Big), \quad w \in X_{r_0}. \tag{2.19}$$

In view of the assumption $\varepsilon r_0^{1-s_c} \ll 1$, the continuity estimate on the resolvent (2.6) and the nonlinear estimates (2.17), (2.18), the Banach fixed point theorem applies and yields a unique solution w to (2.19) with

$$\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.$$

Differentiating (2.19) in space, we immediately infer

$$\|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.$$

Finally, we compute $w_{|\varepsilon=0}$ and $\partial_\varepsilon w_{|\varepsilon=0}$. In view of (2.19), we have

$$w_{|\varepsilon=0} = 0.$$

Also, we have

$$\partial_\varepsilon w = p(p-1)\mathcal{T}\Big(G[\Phi_*, \psi_1, \varepsilon]w\Big) + \varepsilon p(p-1)\mathcal{T}\Big(\partial_\varepsilon G[\Phi_*, \psi_1, \varepsilon]w\Big)$$

and hence

$$\partial_\varepsilon w_{|\varepsilon=0} = p(p-1)\mathcal{T}\Big(G[\Phi_*,\psi_1,\varepsilon]w\Big)_{|\varepsilon=0}.$$

We have

$$G[\Phi_*,\psi_1,\varepsilon]w_{|\varepsilon=0} \quad = \quad \left(\int_0^1(1-s)\Phi_*^{p-2}ds\right)\psi_1^2 = \frac{1}{2}\Phi_*^{p-2}\psi_1^2$$

which yields

$$\partial_\varepsilon w_{|\varepsilon=0} = \frac{p(p-1)}{2}\mathcal{T}\Big(\Phi_*^{p-2}\psi_1^2\Big).$$

The continuity estimate (2.6) and the asymptotic behavior of ψ_1 (2.3) (2.4) yield

$$\|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}.$$

step 2 Proof of the nonlinear estimates (2.17), (2.18). Note first that in view of Lemma 2.1 and the definition of $\|\cdot\|_{X_{r_0}}$, we have for $r_0\le r\le 1$,

$$|w(r)|+|\psi_1(r)| \lesssim r^{-\frac12} = r^{1-\frac{2}{p-1}-s_c} \lesssim r^{1-s_c}|\Phi_*(r)| \le r_0^{1-s_c}|\Phi_*(r)|$$

while for $r\ge 1$, we have

$$|w(r)|+|\psi_1(r)| \lesssim |\Phi_*(r)|,$$

and hence, our choice of ε yields for all $r\ge r_0$

$$\varepsilon|\psi_1(r)|+\varepsilon|w(r)| \lesssim |\Phi_*(r)|.$$

Next, we estimate $G[\Phi_*,\psi_1,\varepsilon]w$. For $r_0\le r\le 1$, we have

$$\begin{aligned} & |G[\Phi_*,\psi_1,\varepsilon]w| \le (|\Phi_*(r)|+\varepsilon(|\psi_1(r)|+|w(r)|))^{p-2}(|\psi_1(r)|+|w(r)|)^2 \\ \lesssim \;& |\Phi_*(r)|^{p-2}(|\psi_1(r)|+|w(r)|)^2 \lesssim \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2}\left(\frac{1}{r^{\frac12}}\right)^2(1+\|w\|_{X_{r_0}})^2 \lesssim r^{\frac{2}{p-1}-3} \end{aligned}$$

and hence

$$\int_{r_0}^1|G[\Phi_*,\psi_1,\varepsilon]w|r'^{\frac32}dr' \lesssim \left(\int_{r_0}^1 r'^{-s_c}dr'\right) \lesssim r_0^{1-s_c}.$$

Also, for $r\ge 1$, we have

$$\begin{aligned} |G[\Phi_*,\psi_1,\varepsilon]w| &\le (|\Phi_*(r)|+\varepsilon(|\psi_1(r)|+|w(r)|))^{p-2}(|\psi_1(r)|+|w(r)|)^2 \\ &\lesssim \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^p(1+\|w\|_{X_{r_0}})^2 \lesssim \frac{1}{r^{2+\frac{2}{p-1}}} \end{aligned}$$

and hence

$$\sup_{r\ge1} r^{\frac{2}{p-1}+2}|G[\Phi_*,\psi_1,\varepsilon]w| \quad \lesssim \quad 1$$

and (2.17) is proved. We now prove the contraction estimate:

$$\begin{aligned}
& G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2 \\
= & \left(\int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_1))^{p-2}ds\right)(\psi_1+w_1)^2 \\
- & \left(\int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_2))^{p-2}ds\right)(\psi_1+w_2)^2 \\
= & \left(\int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_1))^{p-2}ds\right)\Big((\psi_1+w_1)^2-(\psi_1+w_2)^2\Big) \\
+ & \Bigg(\int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_1))^{p-2}ds \\
- & \int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_2))^{p-2}ds\Bigg)(\psi_1+w_2)^2 \\
= & \left(\int_0^1 (1-s)(\Phi_*+s\varepsilon(\psi_1+w_1))^{p-2}ds\right)\Big(2\psi_1+w_1+w_2\Big)(w_1-w_2) \\
+ & (p-2)\left(\int_0^1 s(1-s)\int_0^1 (\Phi_*+s\varepsilon(\psi_1+w_1)+\sigma s\varepsilon(w_2-w_1))^{p-3}d\sigma ds\right) \\
\times & (\psi_1+w_2)^2\varepsilon(w_1-w_2)
\end{aligned}$$

and hence

$$\begin{aligned}
& |G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2| \\
\lesssim & (|\Phi_*(r)|+\varepsilon(|\psi_1(r)|+|w_1(r)|))^{p-2}\Big(|\psi_1(r)|+|w_1(r)|+|w_2(r)|\Big)|w_1(r)-w_2(r)| \\
& +(|\Phi_*(r)|+\varepsilon(|\psi_1(r)|+|w_1(r)|))^{p-3}(|\psi_1(r)|+|w_2(r)|)^2\varepsilon|w_1(r)-w_2(r)| \\
\lesssim & \left\{|\Phi_*(r)|^{p-2}(|\psi_1(r)|+|w_1(r)|+|w_2(r)|)+\varepsilon|\Phi_*(r)|^{p-3}(|\psi_1(r)|+|w_2(r)|)^2\right\}|w_1(r)-w_2(r)|.
\end{aligned}$$

For $r_0\le r\le 1$, we have

$$\begin{aligned}
& \Big|G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2\Big| \\
\lesssim & \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2}\left(\frac{1}{r^{\frac12}}\right)^2(1+\|w_1\|_{X_{r_0}}+\|w_2\|_{X_{r_0}})\|w_1-w_2\|_{X_{r_0}} \\
+ & \varepsilon\left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-3}\left(\frac{1}{r^{\frac12}}\right)^3(1+\|w_1\|_{X_{r_0}}+\|w_2\|_{X_{r_0}})^2\|w_1-w_2\|_{X_{r_0}} \\
\lesssim & \left(r^{\frac{2}{p-1}-3}+\varepsilon r^{\frac{4}{p-1}-\frac72}\right)\|w_1-w_2\|_{X_{r_0}}
\end{aligned}$$

and hence

$$\begin{aligned}
& \int_{r_0}^1 |G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2|r'^{\frac32}dr' \\
\lesssim & \left(\int_{r_0}^1 r'^{-s_c}dr'+\varepsilon\int_{r_0}^1 r'^{1-2s_c}dr'\right)\|w_1-w_2\|_{X_{r_0}} \\
\lesssim & r_0^{1-s_c}(1+\varepsilon r_0^{1-s_c})\|w_1-w_2\|_{X_{r_0}}\lesssim r_0^{1-s_c}\|w_1-w_2\|_{X_{r_0}}.
\end{aligned}$$

Similarly, for $r \geq 1$,

$$\begin{aligned} & |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\ \lesssim \ & \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^p (1 + \|w_1\|_{X_{r_0}} + \|w_2\|_{X_{r_0}})^3 \|w_1 - w_2\|_{X_{r_0}} \\ \lesssim \ & \frac{1}{r^{2+\frac{2}{p-1}}} \|w_1 - w_2\|_{X_{r_0}} \end{aligned}$$

and hence

$$\sup_{r \geq 1} r^{\frac{2}{p-1}+2} |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \lesssim \|w_1 - w_2\|_{X_{r_0}}.$$

This concludes the proof of (2.17), (2.18) and of Proposition 2.2. □

2.2. **Constructing interior self-similar solutions.** We now construct the family of inner solutions to (2.1) in $[0, r_0]$ which after renormalization bifurcate from the *stationary* equation and the ground state solution Q.

We start with the continuity of the resolvent of the linearized operator H close to Q in suitable weighted spaces. Given $r_1 \gg 1$, we define Y_{r_1} as the space of functions on $(0, r_1)$ such that the following norm is finite

$$\|w\|_{Y_{r_1}} = \sup_{0 \leq r \leq r_1} (1+r)^{-\frac{3}{2}} (|w| + r|\partial_r w|).$$

Lemma 2.3 (Interior resolvent of H)**.** 1. Basis of fundamental solutions*: we have*

$$H(\Lambda Q) = 0, \quad H\rho = 0$$

with the following asymptotic behavior as $r \to +\infty$

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right), \ \rho(r) = \frac{c_9 \sin(\omega \log(r) + c_{10})}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right),$$

where $c_7, c_9 \neq 0$*,* $c_8, c_{10} \in \mathbb{R}$*.*
2. Continuity of the resolvent*: let the inverse*

$$\mathcal{S}(f) = \left(\int_0^r f \rho r'^2 dr'\right) \Lambda Q - \left(\int_0^r f \Lambda Q r'^2 dr'\right) \rho$$

then

$$\|\mathcal{S}(f)\|_{Y_{r_1}} \lesssim \sup_{0 \leq r \leq r_1} (1+r)^{\frac{1}{2}} |f|. \tag{2.20}$$

Proof. **step 1** Fundamental solutions. Define

$$Q_\lambda(r) = \lambda^{\frac{2}{p-1}} Q(\lambda r), \quad \lambda > 0,$$

then

$$\Delta Q_\lambda + Q_\lambda^p = 0 \text{ for all } \lambda > 0$$

and differentiating w.r.t. λ and evaluating at $\lambda = 1$ yields

$$H(\Lambda Q) = 0.$$

Let ρ be another solution to $H(\rho) = 0$ which does not depend linearly on ΛQ, we aim at deriving the asymptotic of both ΛQ and ρ as $r \to +\infty$.
Limiting problem We first solve

$$-\partial_r^2 \varphi - \frac{2}{r} \partial_r \varphi - \frac{p c_\infty^{p-1}}{r^2} \varphi = f. \tag{2.21}$$

The homogeneous problem admits the explicit basis of solutions

$$\varphi_1 = \frac{\sin(\omega\log(r))}{r^{\frac{1}{2}}}, \quad \varphi_2 = \frac{\cos(\omega\log(r))}{r^{\frac{1}{2}}}, \tag{2.22}$$

and the corresponding Wronskian is given by

$$W(r) = \varphi_1'(r)\varphi_2(r) - \varphi_2'(r)\varphi_1(r) = \frac{\omega}{r^2}.$$

Using the variation of constants, the solutions to (2.21) are given by

$$\varphi(r) = \left(a_{1,0} + \int_r^{+\infty} f\varphi_2 \frac{r'^2}{\omega}dr'\right)\varphi_1 + \left(a_{2,0} - \int_r^{+\infty} f\varphi_1 \frac{r'^2}{\omega}dr'\right)\varphi_2.$$

Inverting H. We now claim that all solutions to $H(\phi) = 0$ admit an expansion

$$\phi(r) = a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \text{ as } r \to +\infty. \tag{2.23}$$

Indeed, we rewrite the equation

$$-\partial_r^2\phi - \frac{2}{r}\partial_r\phi - \frac{pc_\infty^{p-1}}{r^2}\phi = f, \quad f = p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\phi(r),$$

and hence

$$\phi = a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}, \quad \widetilde{\phi} = \mathcal{F}\left(\widetilde{\phi}\right) \tag{2.24}$$

where

$$\begin{aligned}
\mathcal{F}\left(\widetilde{\phi}\right)(r) \;=\; & -\left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}\right)(r')\varphi_2\frac{r'^2}{\omega}dr'\right)\varphi_1 \\
& + \left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}\right)(r')\varphi_1\frac{r'^2}{\omega}dr'\right)\varphi_2.
\end{aligned}$$

Recall that

$$Q(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}} + O\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty$$

so that

$$\left|p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\right| \lesssim \frac{1}{r^{1+s_c}} \text{ for } r \geq 1.$$

We infer for $r \geq 1$

$$\begin{aligned}
\left|\mathcal{F}\left(\widetilde{\phi}\right)(r)\right| &\lesssim \frac{1}{r^{\frac{1}{2}}}\left(\int_r^{+\infty}\left(\frac{1}{r'^{s_c}} + \frac{1}{r'^{s_c - \frac{1}{2}}}\left|\widetilde{\phi}\right|(r')\right)dr'\right) \\
&\lesssim \frac{1}{r^{s_c - \frac{1}{2}}} + \frac{1}{r^{\frac{1}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c - \frac{1}{2}}}\left|\widetilde{\phi}\right|(r')dr'\right)
\end{aligned}$$

and

$$\left|\mathcal{F}\left(\widetilde{\phi}_1\right)(r) - \mathcal{F}\left(\widetilde{\phi}_2\right)(r)\right| \lesssim \frac{1}{r^{\frac{1}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c - \frac{1}{2}}}\left|\widetilde{\phi}_1 - \widetilde{\phi}_2\right|(r')dr'\right).$$

Thus, for $R \geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r \geq R} r^{s_c - \frac{1}{2}}\left|\widetilde{\phi}\right|(r)$$

and yields a unique solution $\widetilde{\phi}$ to (2.24) with

$$\sup_{r\geq R} r^{s_c-\frac{1}{2}}\left|\widetilde{\phi}\right|(r)\leq 1,$$

and (2.23) is proved.
In particular, in view of the explicit formula (2.22) for φ_1 and φ_2, and in view of the fact that $H(\Lambda Q)=0$ and $H(\rho)=0$, we infer as $r\to+\infty$

$$\Lambda Q(r)=\frac{c_7\sin(\omega\log(r)+c_8)}{r^{\frac{1}{2}}}+O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right),\ \rho=\frac{c_9\sin(\omega\log(r)+c_{10})}{r^{\frac{1}{2}}}+O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right) \tag{2.25}$$

where $c_7,c_9\neq 0$, $c_8,c_{10}\in\mathbb{R}$.

step 2 Continuity of the resolvent. We compute

$$W:=\Lambda Q'\rho-\rho'\Lambda Q,\quad W'=-\frac{2}{r}W,\quad W=\frac{-1}{r^2},$$

without loss of generality. Still without loss of generality for $R_0>0$ small enough such that $\Lambda Q>0$ on $[0,R_0]$ the integration of the Wronskian law yields

$$\rho=-\Lambda Q\int_r^{R_0}\frac{1}{(\Lambda Q)^2r'^2}dr'$$

on $(0,R_0]$ which ensures

$$|\rho(r)|\lesssim\frac{1}{r},\ |\partial_r\rho(r)|\lesssim\frac{1}{r^2}\ \text{as}\ \ r\to 0. \tag{2.26}$$

We now solve

$$H(w)=f,$$

using the variation of constants which yields

$$w\ \ =\ \ \left(a_1+\int_0^r f\rho r'^2dr'\right)\Lambda Q+\left(a_2-\int_0^r f\Lambda Q r'^2dr'\right)\rho.$$

In particular, $\mathcal{S}(f)$ corresponds to the choice $a_1=a_2=0$ and thus

$$H(\mathcal{S}(f))=f.$$

Finally, using the estimates (2.25), (2.26), we estimate for $0\leq r\leq 1$:

$$\begin{aligned}|\mathcal{S}(f)|&=\left|\left(\int_0^r f\rho r'^2dr'\right)\Lambda Q-\left(\int_0^r f\Lambda Q r'^2dr'\right)\rho\right|\\ &\lesssim\ \left(\int_0^r r'dr'+\frac{1}{r}\int_0^r r'^2dr'\right)\sup_{0\leq r\leq 1}|f|\lesssim\sup_{0\leq r\leq r_1}(1+r)^{\frac{1}{2}}|f|,\end{aligned}$$

$$\begin{aligned}|r\partial_r\mathcal{S}(f)|&=\left|\left(\int_0^r f\rho r'^2dr'\right)r\partial_r\Lambda Q-\left(\int_0^r f\Lambda Q r'^2dr'\right)r\partial_r\rho\right|\\ &\lesssim\ \left(r^2\int_0^r r'dr'+\frac{1}{r}\int_0^r r'^2dr'\right)\sup_{0\leq r\leq 1}|f|\lesssim\sup_{0\leq r\leq r_1}(1+r)^{\frac{1}{2}}|f|,\end{aligned}$$

and for $1\le r\le r_1$:

$$
\begin{aligned}
&(1+r)^{-\frac32}|\mathcal{S}(f)| = (1+r)^{-\frac32}\left|\left(\int_0^r f\rho r'^2dr'\right)\Lambda Q-\left(\int_0^r f\Lambda Q r'^2dr'\right)\rho\right|\\
\lesssim\ & (1+r)^{-2}\left(\int_0^r f(1+r')^{\frac32}dr'\right)\lesssim (1+r)^{-2}\left(\int_0^r (1+r')dr'\right)\sup_{0\le r\le r_1}(1+r)^{\frac12}|f|\\
\lesssim\ & \sup_{0\le r\le r_1}(1+r)^{\frac12}|f|
\end{aligned}
$$

$$
\begin{aligned}
&(1+r)^{-\frac32}|r\partial_r\mathcal{S}(f)| = (1+r)^{-\frac32}\left|\left(\int_0^r f\rho r'^2dr'\right)r\partial_r\Lambda Q-\left(\int_0^r f\Lambda Q r'^2dr'\right)r\partial_r\rho\right|\\
\lesssim\ & (1+r)^{-2}\left(\int_0^r f(1+r')^{\frac32}dr'\right)\lesssim (1+r)^{-2}\left(\int_0^r (1+r')dr'\right)\sup_{0\le r\le r_1}(1+r)^{\frac12}|f|\\
\lesssim\ & \sup_{0\le r\le r_1}(1+r)^{\frac12}|f|,
\end{aligned}
$$

which concludes the proof of (2.20) and Lemma 2.3. □

We are now in position to build the family of interior solutions:

Proposition 2.4 (Construction of the interior solution)**.** *Let $r_0>0$ small enough and let $0<\lambda\le r_0$. Then, there exists a solution u to*

$$\Delta u-\Lambda u+u^p=0 \ \text{on}\ 0\le r\le r_0$$

of the form

$$u=\frac{1}{\lambda^{\frac{2}{p-1}}}(Q+\lambda^2T_1)\left(\frac{r}{\lambda}\right)$$

with

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}}+\|\Lambda T_1\|_{Y_{\frac{r_0}{\lambda}}}+\|\Lambda^2T_1\|_{Y_{\frac{r_0}{\lambda}}}\lesssim 1. \tag{2.27}$$

Proof. This is again a classical consequence of Lemma 2.3.

step 1 Setting up the Banach fixed point. We look for u of the form

$$u=\frac{1}{\lambda^{\frac{2}{p-1}}}(Q+\lambda^2T_1)\left(\frac{r}{\lambda}\right)$$

so that u solves $\Delta u-\Lambda u+u^p=0$ on $[0,r_0]$ if and only if

$$H(T_1)=J[Q,\lambda^2]T_1 \ \text{on}\ 0\le r\le r_1$$

where

$$r_1=\frac{r_0}{\lambda}\ge 1$$

so that

$$\lambda^2r_1^2=r_0^2\ll 1$$

and with

$$J[Q,\lambda^2]T_1 \ = \ -\Lambda Q-\lambda^2\Lambda T_1+p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2T_1)^{p-2}ds\right)T_1^2.$$

We claim the nonlinear estimates: assume $\|w\|_{Y_{r_1}}\lesssim 1$, then

$$\sup_{0\le r\le r_1}(1+r)^{\frac12}|J[Q,\lambda^2]w|\lesssim 1, \tag{2.28}$$

$$\sup_{0\le r\le r_1}(1+r)^{\frac12}|J[Q,\lambda^2]w_1-J[Q,\lambda^2]w_2|\lesssim r_1^2\lambda^2\|w_1-w_2\|_{Y_{r_1}}. \tag{2.29}$$

Assume (2.28), (2.29), we then look for T_1 as the solution to the fixed point

$$T_1 = \mathcal{S}(J[Q,\lambda^2]T_1). \tag{2.30}$$

In view of the bound $\lambda^2 r_1^2 \ll 1$, the resolvent estimate (2.20) and the nonlinear estimates (2.28), (2.29), the Banach fixed point theorem applies and yields a unique solution T_1 to (2.30) which furthermore satisfies:

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}} \lesssim 1.$$

step 2 Proof of (2.28), (2.29). Note first that for $0 \le r \le r_1$, we have

$$|w(r)| \lesssim (1+r)^{\frac{3}{2}} = r_1^2(1+r)^{-\frac{1}{2}} \lesssim r_1^2|Q(r)|.$$

Thus, we infer for all $0 \le r \le r_1$

$$\lambda^2|w(r)| \lesssim \lambda^2 r_1^2|Q(r)|$$

and hence, our choice of λ yields for all $0 \le r \le r_1$

$$\lambda^2|w(r)| \lesssim |Q(r)|.$$

Next, we estimate $J[Q,\lambda^2]w$. For $0 \le r \le r_1$, we have

$$\begin{aligned}
&|J[Q,\lambda^2]w| \le |\Lambda Q| + p(p-1)\lambda^2(|Q|+\lambda^2|w|)^{p-2}|w|^2 + \lambda^2\left|\frac{1}{2}w + r\partial_r w\right| \\
\lesssim\ & |\Lambda Q| + \lambda^2|Q|^{p-2}|w|^2 + \lambda^2|\frac{1}{2}w + r\partial_r w| \\
\lesssim\ & (1+r)^{-\frac{1}{2}} + \lambda^2(1+r)^{-\frac{2(p-2)}{p-1}}(1+r)^3\|w\|_{Y_{r_1}}^2 + \lambda^2(1+r)^{\frac{3}{2}}\|w\|_{Y_{r_1}}^2 \\
\lesssim\ & (1+r)^{-\frac{1}{2}}\left(1+\lambda^2(1+r)^{\frac{2}{p-1}+\frac{3}{2}} + \lambda^2(1+r)^2\right) \\
\lesssim\ & (1+r)^{-\frac{1}{2}}\left(1+\lambda^2(1+r)^{-s_c+3} + \lambda^2 r_1^2\right) \lesssim (1+r)^{-\frac{1}{2}}\left(1+\lambda^2 r_1^2\right) \lesssim (1+r)^{-\frac{1}{2}}
\end{aligned}$$

and hence

$$\sup_{0\le r\le r_1}(1+r)^{\frac{1}{2}}|J[Q,\lambda^2]w| \quad \lesssim 1.$$

Next, we estimate $|J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2|$. We have

$$\begin{aligned}
& J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2\\
= \;& p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2 w_1)^{p-2}ds\right)w_1^2 - p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2 w_2)^{p-2}ds\right)w_2^2\\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2)+r(\partial_r w_1-\partial_r w_2)\right)w\\
= \;& p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2 w_1)^{p-2}ds\right)(w_1^2-w_2^2)\\
+ \;& p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2 w_1)^{p-2}ds - \int_0^1(1-s)(Q+s\lambda^2 w_2)^{p-2}ds\right)w_2^2\\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2)+r(\partial_r w_1-\partial_r w_2)\right)w\\
= \;& p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2 w_1)^{p-2}ds\right)(w_1+w_2)(w_1-w_2)\\
+ \;& p(p-1)(p-2)\lambda^4\left(\int_0^1 s(1-s)\int_0^1(Q+s\lambda^2 w_1+\sigma s\lambda^2(w_2-w_1))^{p-3}d\sigma ds\right)w_2^2(w_1-w_2)\\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2)+r(\partial_r w_1-\partial_r w_2)\right)w
\end{aligned}$$

and hence

$$\begin{aligned}
& |J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2| \lesssim \lambda^2(|Q(r)|+\lambda^2|w_1(r)|)^{p-2}(|w_1(r)|+|w_2(r)|)|w_1(r)-w_2(r)|\\
& +\lambda^4(|Q(r)|+\lambda^2|w_1(r)|+\lambda^2|w_2(r)|)^{p-3}|w_2(r)|^2|w_1(r)-w_2(r)|\\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2)+r(\partial_r w_1-\partial_r w_2)\right)w\\
\lesssim \;& \lambda^2|Q(r)|^{p-2}(|w_1(r)|+|w_2(r)|)|w_1(r)-w_2(r)| + \lambda^4|Q(r)|^{p-3}|w_2(r)|^2|w_1(r)-w_2(r)|\\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2)+r(\partial_r w_1-\partial_r w_2)\right)w.
\end{aligned}$$

This yields

$$\begin{aligned}
& |J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2| \lesssim \lambda^2(1+r)^{-\frac{2(p-2)}{p-1}}(1+r)^3(\|w_1\|_{Y_{r_1}}+\|w_2\|_{Y_{r_1}})\|w_1-w_2\|_{Y_{r_1}}\\
& \lambda^4(1+r)^{-\frac{2(p-3)}{p-1}}(1+r)^{\frac{9}{2}}\|w_2\|_{Y_{r_1}}^2\|w_1-w_2\|_{Y_{r_1}} + \lambda^2(1+r)^{\frac{3}{2}}\|w_1-w_2\|_{Y_{r_1}}\\
\lesssim \;& \lambda^2(1+r)^{-\frac{1}{2}}\left((1+r)^{\frac{2}{p-1}+\frac{3}{2}}+\lambda^2(1+r)^{\frac{4}{p-1}+3}+(1+r)^2\right)\|w_1-w_2\|_{Y_{r_1}}\\
\lesssim \;& \lambda^2(1+r)^{-\frac{1}{2}}\left((1+r)^{-s_c+3}+\lambda^2(1+r)^{-2s_c+6}+(1+r)^2\right)\|w_1-w_2\|_{Y_{r_1}}\\
\lesssim \;& r_1^2\lambda^2(1+r)^{-\frac{1}{2}}\left(1+\lambda^2 r_1^2\right)\|w_1-w_2\|_{Y_{r_1}} \lesssim r_1^2\lambda^2(1+r)^{-\frac{1}{2}}\|w_1-w_2\|_{Y_{r_1}}
\end{aligned}$$

which concludes the proof of (2.29) and Proposition 2.4. □

2.3. **The matching.** We now construct a solution to (2.1) by matching the exterior solution to (2.1) constructed in section 2.1 on $[r_0,+\infty)$ to the interior solution to (2.1) constructed in section 2.2 on $[0,r_0]$. The oscillations (1.17) allow to perform the matching at r_0 for a quantized sequence of the small parameter ε introduced in Proposition 2.2.

Proposition 2.5 (Existence of a countable number of smooth selfsimilar profiles)**.** *There exists $N \in \mathbb{N}$ large enough so that for all $n \geq N$, there exists a smooth solution Φ_n to* (2.1) *such that $\Lambda\Phi_n$ vanishes exactly n times.*

Proof. **step 1** Initialization. Since

$$\psi_1(r) = \frac{c_3 \sin(\omega \log(r) + c_4)}{r^{\frac{1}{2}}} + O\left(r^{\frac{3}{2}}\right) \text{ as } r \to 0, \quad c_3 \neq 0$$

we compute

$$\Lambda\psi_1(r) = c_3 \frac{(1-s_c)\sin(\omega \log(r) + c_4) + \omega \cos(\omega \log(r) + c_4)}{r^{\frac{1}{2}}} + O\left(r^{\frac{3}{2}}\right) \text{ as } r \to 0.$$

We may therefore choose $0 < r_0 \ll 1$ such that

$$\psi_1(r_0) = \frac{c_3}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right), \quad \Lambda\psi_1(r_0) = \frac{c_3(1-s_c)}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right), \tag{2.31}$$

and Proposition 2.2 and Proposition 2.4 apply. We therefore choose ε and λ such that

$$0 < \varepsilon \ll r_0^{s_c - 1}, \quad 0 < \lambda \leq r_0,$$

and have from Proposition 2.2 an exterior solution u_{ext} to

$$-\Delta u_{ext} + \Lambda u_{ext} - u_{ext}^p, \quad r \geq r_0$$

such that

$$u_{ext}[\varepsilon] = \Phi_* + \varepsilon\psi_1 + \varepsilon w$$

and

$$\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}, \quad \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}. \tag{2.32}$$

We also have from Proposition 2.4 an interior solution u_{int} to

$$-\Delta u_{int} + \Lambda u_{int} - u_{int}^p, \quad 0 \leq r \leq r_0$$

such that

$$u_{int}[\lambda] \quad = \quad \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + \lambda^2 T_1)\left(\frac{r}{\lambda}\right).$$

with

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}} \lesssim 1. \tag{2.33}$$

We now would like to match the two solutions at $r = r_0$ which is equivalent to requiring that

$$u_{ext}(r_0) - u_{int}(r_0) = 0 \text{ and } u'_{ext}(r_0) - u'_{int}(r_0).$$

step 2 Matching the functions. We introduce the map

$$\mathcal{F}[r_0](\varepsilon, \lambda) := u_{ext}[\varepsilon](r_0) - u_{int}[\lambda](r_0).$$

We compute

$$\partial_\varepsilon \mathcal{F}[r_0](\varepsilon, \lambda) \quad = \quad \partial_\varepsilon u_{ext}[\varepsilon](r_0) = \psi_1(r_0) + w(r_0) + \varepsilon \partial_\varepsilon w(r_0).$$

In particular, since $w_{|\varepsilon=0} = 0$ and $\|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}$ in view of Proposition 2.2, we have

$$\partial_\varepsilon \mathcal{F}[r_0](0,0) = \psi_1(r_0) \neq 0$$

since we assumed that $\psi_1(r_0) \neq 0$. Also, in view of the asymptotic behavior of Q at infinity, we have as $\lambda \to 0_+$

$$\left|\frac{1}{\lambda^{\frac{2}{p-1}}}(Q - \Phi_* + \lambda^2 T_1)\left(\frac{r_0}{\lambda}\right)\right| \quad \lesssim \quad \frac{1}{\lambda^{\frac{2}{p-1}}}\left(\frac{1}{r^{\frac{1}{2}}} + \frac{\lambda^2 r^2}{r^{\frac{1}{2}}}\right)\left(\frac{r_0}{\lambda}\right) \lesssim \frac{\lambda^{\frac{1}{2} - \frac{2}{p-1}}}{r_0^{\frac{1}{2}}} \lesssim \frac{\lambda^{s_c - 1}}{r_0^{\frac{1}{2}}}$$

and hence, since $s_c>1$, we infer

$$\lim_{\lambda\to 0_+}\frac{1}{\lambda^{\frac{2}{p-1}}}(Q-\Phi_*+\lambda^2T_1)\left(\frac{r_0}{\lambda}\right)=0.$$

Since

$$\frac{1}{\lambda^{\frac{2}{p-1}}}\Phi_*\left(\frac{r_0}{\lambda}\right)=\Phi_*(r_0),$$

this yields

$$\mathcal{F}[r_0](0,0)=\Phi_*(r_0)-\Phi_*(r_0)=0.$$

We may thus apply the implicit function theorem[12] which yields the existence of $\lambda_0>0$ and a $\mathcal{C}^{\min(1,(s_c-1)_-)}$ function $\varepsilon(\lambda)$ defined on $[0,\lambda_0)$ such that $\mathcal{F}(\varepsilon(\lambda),\lambda)=0$ and hence

$$u_{ext}[\varepsilon(\lambda)](r_0)=u_{int}[\lambda](r_0)\ \text{ on } [0,\lambda_0).$$

step 3 Control of $\varepsilon(\lambda)$. We claim for $\lambda\in[0,\lambda_0)$

$$\varepsilon(\lambda)=\frac{1}{\psi_1(r_0)\lambda^{\frac{2}{p-1}}}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)+O\left[\lambda^{s_c-1}(r_0^2+\lambda^{s_c-1}r_0^{1-s_c})\right]. \tag{2.34}$$

Indeed, by construction

$$u_{ext}[\varepsilon(\lambda)](r_0)=u_{int}[\lambda](r_0)$$

which is equivalent to

$$\varepsilon(\lambda)\psi_1(r_0)+\varepsilon(\lambda)w(r_0)=\frac{1}{\lambda^{\frac{2}{p-1}}}(Q-\Phi_*+\lambda^2T_1)\left(\frac{r_0}{\lambda}\right). \tag{2.35}$$

We infer from (2.31), (2.32), (2.33) and the asymptotic of Q:

$$\varepsilon(\lambda)\psi_1(r_0)+\varepsilon(\lambda)w(r_0)=\varepsilon(\lambda)\frac{c_3}{r_0^{\frac12}}\left(1+O(r_0^2)+O(\varepsilon(\lambda)r_0^{1-s_c})\right),$$

$$\frac{1}{\lambda^{\frac{2}{p-1}}}\left|Q-\Phi_*+\lambda^2T_1)\left(\frac{r_0}{\lambda}\right)\right|\lesssim\frac{\lambda^{s_c-1}}{r_0^{\frac12}}(1+O(r_0^2)).$$

This first yields using (2.14)

$$|\varepsilon(\lambda)|\lesssim\lambda^{s_c-1}. \tag{2.36}$$

which reinjected into (2.35) yields (2.34).

step 4 Computation of the spatial derivatives. We consider the difference of spatial derivatives at r_0 for $\lambda\in[0,\lambda_0)$

$$\mathcal{G}[r_0](\lambda):=u_{ext}[\varepsilon(\lambda)]'(r_0)-u_{int}[\lambda]'(r_0)$$

and claim the leading order expansion:

$$\begin{aligned}\mathcal{G}[r_0](\lambda)\ &=\ \lambda^{s_c-1}\left[\frac{c_1c_3\omega}{\psi_1(r_0)r_0^2}\sin\left(-\omega\log(\lambda)+c_2-c_4\right)\right.\\ &+\ \left.O\left(r_0^{-s_c-\frac12}\lambda^{s_c-1}+r_0^{\frac12}\right)\right].\end{aligned} \tag{2.37}$$

[12] We actually apply the implicit function theorem to

$$\widetilde{\mathcal{F}}[r_0](\varepsilon,\mu):=\mathcal{F}(\varepsilon,\mu^{\frac{1}{s_c-1-\delta}})$$

for any $0<\delta<s_c-1$ so that $\widetilde{\mathcal{F}}\in\mathcal{C}^1$. This yields the existence of $\widetilde{\varepsilon}\in\mathcal{C}^1$ and we choose $\varepsilon(\lambda)=\widetilde{\varepsilon}(\lambda^{s_c-1-\delta})$ so that ε belongs indeed to $\mathcal{C}^{\min(1,(s_c-1)_-)}$.

Indeed,

$$\mathcal{G}[r_0](\lambda) \quad = \quad \varepsilon(\lambda)\psi_1'(r_0)+\varepsilon(\lambda)w'(r_0)-\frac{1}{\lambda^{\frac{2}{p-1}+1}}(Q'-\Phi_*'+\lambda^2T_1')\left(\frac{r_0}{\lambda}\right).$$

From (2.36), (2.16):

$$|\varepsilon(\lambda)w'(r_0)| \quad \lesssim \quad \lambda^{s_c-1}|w'(r_0)|\lesssim \lambda^{2(s_c-1)}r_0^{-\frac{1}{2}-s_c}$$

and from (2.27)

$$\left|\frac{1}{\lambda^{\frac{2}{p-1}+1}}\lambda^2T_1'\left(\frac{r_0}{\lambda}\right)\right| \quad \lesssim \quad r_0^{\frac{1}{2}}\lambda^{s_c-1}$$

and hence using (2.34), (2.31):

$$\begin{aligned}
\mathcal{G}[r_0](\lambda) &= \varepsilon(\lambda)\psi_1'(r_0)-\lambda^{s_c-1}\frac{1}{\lambda^{\frac{3}{2}}}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)+O\left(\left(r_0^{-\frac{3}{2}}\lambda^{s_c-1}+r_0^{\frac{1}{2}}\right)\lambda^{s_c-1}\right)\\
&= \lambda^{s_c-1}\left(\frac{1}{\lambda^{\frac{1}{2}}\psi_1(r_0)}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0)-\frac{1}{\lambda^{\frac{3}{2}}}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\right)\\
&\quad +O\left(\left(r_0^{-s_c-\frac{1}{2}}\lambda^{s_c-1}+r_0^{\frac{1}{2}}\right)\lambda^{s_c-1}\right)\\
&= \frac{1}{r_0^{\frac{1}{2}}\psi_1(r_0)}\lambda^{s_c-1}\left\{\left(\frac{r_0}{\lambda}\right)^{\frac{1}{2}}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0)-\left(\frac{r_0}{\lambda}\right)^{\frac{3}{2}}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{\psi_1(r_0)}{r_0}\right\}\\
&\quad +O\left(\left(r_0^{-s_c-\frac{1}{2}}\lambda^{s_c-1}+r_0^{\frac{1}{2}}\right)\lambda^{s_c-1}\right).
\end{aligned}$$

Recall that

$$\begin{aligned}
&\psi_1(r)=\frac{c_3\sin(\omega\log(r)+c_4)}{r^{\frac{1}{2}}}+O\left(r^{\frac{3}{2}}\right)\ \text{ as } r\to 0,\\
&\psi_1'(r)=-\frac{c_3\sin(\omega\log(r)+c_4)}{2r^{\frac{3}{2}}}+\frac{c_3\omega\cos(\omega\log(r)+c_4)}{r^{\frac{3}{2}}}+O\left(r^{\frac{1}{2}}\right)\ \text{ as } r\to 0,\\
&Q(r)-\Phi_*(r)=\frac{c_1\sin\left(\omega\log(r)+c_2\right)}{r^{\frac{1}{2}}}+O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right)\ \text{ as } r\to +\infty,\\
&Q'(r)-\Phi_*'(r)=-\frac{c_1\sin\left(\omega\log(r)+c_2\right)}{2r^{\frac{3}{2}}}+\frac{c_1\omega\cos\left(\omega\log(r)+c_2\right)}{r^{\frac{3}{2}}}+O\left(\frac{1}{r^{s_c+\frac{1}{2}}}\right)\ \text{ as } r\to +\infty
\end{aligned}$$

and hence:

$$
\begin{aligned}
&\left(\frac{r_0}{\lambda}\right)^{\frac12}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0)-\left(\frac{r_0}{\lambda}\right)^{\frac32}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{\psi_1(r_0)}{r_0}\\
= \ & \frac{c_1c_3}{r_0^{\frac32}}\left(\sin\left(\omega\log(r_0)-\omega\log(\lambda)+c_2\right)\left(-\frac{\sin(\omega\log(r_0)+c_4)}{2}+\omega\cos(\omega\log(r_0)+c_4)\right)\right)\\
&-\left(-\frac{\sin\left(\omega\log(r_0)-\omega\log(\lambda)+c_2\right)}{2}+\omega\cos\left(\omega\log(r_0)-\omega\log(\lambda)+c_2\right)\right)\sin(\omega\log(r_0)+c_4)\\
&+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right)\\
= \ & \frac{c_1c_3\omega}{r_0^{\frac32}}\Bigg(\sin\left(\omega\log(r_0)-\omega\log(\lambda)+c_2\right)\cos(\omega\log(r_0)+c_4)\\
&-\cos\left(\omega\log(r_0)-\omega\log(\lambda)+c_2\right)\sin(\omega\log(r_0)+c_4)\Bigg)+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right)\\
= \ & \frac{c_1c_3\omega}{r_0^{\frac32}}\sin\left(-\omega\log(\lambda)+c_2-c_4\right)+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right).
\end{aligned}
$$

The collection of above bounds and (2.31) yields (2.37).

step 5 Discrete matching. For $\delta_0>0$ a small enough universal constant such that $\delta_0\geq r_0$ to be chosen later, we consider

$$
\lambda_{k,+}=\exp\left(\frac{-k\pi-c_4+c_2+\delta_0}{\omega}\right),\quad \lambda_{k,-}=\exp\left(\frac{-k\pi-c_4+c_2-\delta_0}{\omega}\right). \tag{2.38}
$$

From

$$
\lim_{k\to+\infty}\lambda_{k,\pm}=0,
$$

there holds for $k\geq k_0$ large enough:

$$
0<\dots<\lambda_{k,+}<\lambda_{k,-}<\dots<\lambda_{k_0,+}<\lambda_{k_0,-}\leq\lambda_0
$$

With the above definition of $\lambda_{k,\pm}$, we have for all $k\geq k_0$

$$
\sin\left(-\omega\log(\lambda_{k,+})+c_2-c_4\right)=(-1)^k\sin(\delta_0),\quad \sin\left(-\omega\log(\lambda_{k,-})+c_2-c_4\right)=-(-1)^k\sin(\delta_0),
$$

and hence

$$
\mathcal{G}[r_0](\lambda_{k,\pm}) \ = \ \pm(-1)^k\lambda_{k,\pm}^{s_c-1}\left(\frac{c_1c_3\omega}{\psi_1(r_0)r_0^2}\sin(\delta_0)+O\left(r_0^{-s_c-\frac12}\lambda_{k,\pm}^{s_c-1}+r_0^{\frac12}\right)\right).
$$

Since $\delta_0\geq r_0$, this yields for r_0 small enough and for any $k\geq k_0$ large enough:

$$
\mathcal{G}[r_0](\lambda_{k,-})\mathcal{G}[r_0](\lambda_{k,+})<0.
$$

Since the function $\lambda\to\mathcal{G}[r_0](\lambda)$ is continuous, we infer from the mean value theorem applied to the intervals $[\lambda_{k,+},\lambda_{k,-}]$ the existence of μ_k such that

$$
\lambda_{k,+}<\mu_k<\lambda_{k,-}\text{ and }\mathcal{G}[r_0](\mu_k)=0\text{ for all }k\geq k_0.
$$

Finally, for $k\geq k_0$, we have

$$
\mathcal{F}[r_0](\varepsilon(\mu_k),\mu_k)=0\text{ and }\mathcal{G}[r_0](\mu_k)=0
$$

which yields

$$
u_{ext}[\varepsilon(\mu_k)](r_0)=u_{int}[\mu_k](r_0)\text{ and }u_{ext}[\varepsilon(\mu_k)]'(r_0)=u_{int}[\mu_k]'(r_0).
$$

and hence the function

$$u_k(r) := \begin{cases} u_{int}[\mu_k](r) & \text{for } 0 \le r \le r_0, \\ u_{ext}[\varepsilon(\mu_k)](r) & \text{for } r > r_0 \end{cases}$$

is smooth and satisfies (2.1).
The rest of the proof is devoted to counting the number of zeroes of Λu_k and showing that this number is an unambiguous way of counting the number of self similar solutions u_k as $k \to +\infty$.

step 6 Zeroes of $\Lambda u_{ext}[\varepsilon]$. We claim that

$$\Lambda u_{ext}[\varepsilon] \;\text{ has as many zeros as }\; \Lambda\psi_1 \;\text{ on }\; r \ge r_0. \tag{2.39}$$

Indeed, $\Lambda\psi_1 + \Lambda w$ does not vanish on $[R_0, +\infty)$ for R_0 large enough from (2.5) and the uniform bound (2.16). Moreover, $\Lambda\psi_1(r_0) \neq 0$ from the normalization (2.31), and the absolute derivative of $\Lambda\psi_1$ at any of its zeroes is uniformly lower bounded using (2.2), (2.4), and hence the uniform smallness (2.16)

$$\|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c} \ll 1$$

yields the claim.

step 7 Zeroes of $\Lambda u_{int}[\mu_k]$. We now claim that

$$\Lambda u_{int}[\mu_k] \;\text{ has as many zeros as }\; \Lambda Q \;\text{ on }\; 0 \le r \le r_0/\mu_k. \tag{2.40}$$

Indeed, recall that

$$\Lambda u_{int}[\mu_k](r) \quad = \quad \frac{1}{\mu_k^{\frac{2}{p-1}}}(\Lambda Q + \mu_k^2 \Lambda T_1)\left(\frac{r}{\mu_k}\right).$$

We now claim

$$\left(\frac{r_0}{\mu_k}\right)^{\frac{1}{2}} \left|\Lambda Q\left(\frac{r_0}{\mu_k}\right)\right| \gtrsim 1. \tag{2.41}$$

Assume (2.41), then since the zeros of ΛQ are simple, since we have

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \quad \text{as } r \to +\infty,$$

since

$$\|\Lambda T_1\|_{Y_{\frac{r_0}{\mu_k}}} \;=\; \sup_{0 \le r \le \frac{r_0}{\mu_k}} (1+r)^{-\frac{3}{2}} |\Lambda T_1| \lesssim 1$$

so that

$$\sup_{0 \le r \le \frac{r_0}{\mu_k}} (1+r)^{\frac{1}{2}} |\mu_k^2 \Lambda T_1| \lesssim r_0^2,$$

and similarly for $\Lambda^2 T_1$, and since

$$\Lambda Q(0) = \frac{2}{p-1} \neq 0,$$

we conclude that $\Lambda Q + \mu_k^2 \Lambda T_1$ has as many zeros as ΛQ on $0 \le r \le r_0/\mu_k$. We deduce that on $0 \le r \le r_0$, $\Lambda u_{int}[\mu_k]$ has as many zeros as ΛQ on $0 \le r \le r_0/\mu_k$.

Proof of (2.41): Recall that

$$u_{ext}[\varepsilon(\mu_k)](r_0) = u_{int}[\mu_k](r_0) \text{ and } u_{ext}[\varepsilon(\mu_k)]'(r_0) = u_{int}[\mu_k]'(r_0),$$

which implies

$$\Lambda u_{ext}[\varepsilon(\mu_k)](r_0) = \Lambda u_{int}[\mu_k](r_0).$$

This yields using (2.34):

$$\frac{\varepsilon(\mu_k)}{\mu_k^{s_c-1}} = \frac{1}{\psi_1(r_0)\mu_k^{\frac12}}(Q-\Phi_*)\left(\frac{r_0}{\mu_k}\right)+O\Big(\mu_k^{s_c-1}r_0^{s_c-1}+r_0^2\Big)$$

and differentiating (2.35):

$$\frac{\varepsilon(\mu_k)}{\mu_k^{s_c-1}} = \frac{1}{\Lambda\psi_1(r_0)\mu_k^{\frac12}}\Lambda Q\left(\frac{r_0}{\mu_k}\right)+O\Big(\mu_k^{s_c-1}r_0^{s_c-1}+r_0^2\Big).$$

We infer

$$\frac{1}{\psi_1(r_0)\mu_k^{\frac12}}(Q-\Phi_*)\left(\frac{r_0}{\mu_k}\right) = \frac{1}{\Lambda\psi_1(r_0)\mu_k^{\frac12}}\Lambda Q\left(\frac{r_0}{\mu_k}\right)+O\Big(\mu_k^{s_c-1}r_0^{s_c-1}+r_0^2\Big).$$

In view of (2.31) which we recall below

$$\psi_1(r_0)=\frac{c_3}{r_0^{\frac12}}+O\left(r_0^{\frac32}\right),\quad \Lambda\psi_1(r_0)=\frac{c_3(1-s_c)}{r_0^{\frac12}}+O\left(r_0^{\frac32}\right),$$

this yields

$$\left|\left(\frac{r_0}{\mu_k}\right)^{\frac12}(Q-\Phi_*)\left(\frac{r_0}{\mu_k}\right)\right|\le\frac{2}{s_c-1}\left|\left(\frac{r_0}{\mu_k}\right)^{\frac12}\Lambda Q\left(\frac{r_0}{\mu_k}\right)\right|+O\Big(\mu_k^{s_c-1}+r_0^2\Big).\quad (2.42)$$

On the other hand,

$$Q(r)-\Phi_*(r)=\frac{c_1\sin\left(\omega\log(r)+c_2\right)}{r^{\frac12}}+O\left(\frac{1}{r^{s_c-\frac12}}\right)\text{ as } r\to+\infty \quad (2.43)$$

and hence as $r\to+\infty$

$$\begin{aligned}\Lambda Q(r) &= c_1\frac{(1-s_c)\sin(\omega\log(r)+c_2)+\omega\cos(\omega\log(r)+c_2)}{r^{\frac12}}+O\left(\frac{1}{r^{s_c-\frac12}}\right)\\ &= c_1\sqrt{(s_c-1)^2+\omega^2}\,\frac{\sin(\omega\log(r)+c_2+\alpha_0)}{r^{\frac12}}+O\left(\frac{1}{r^{s_c-\frac12}}\right)\end{aligned}\quad (2.44)$$

where

$$\cos(\alpha_0)=\frac{1-s_c}{\sqrt{(s_c-1)^2+\omega^2}},\quad \sin(\alpha_0)=\frac{\omega}{\sqrt{(s_c-1)^2+\omega^2}},\quad \alpha_0\in\left(\frac{\pi}{2},\pi\right).$$

Thus there exists $r_2>0$ sufficiently small and a constant $\delta_1>0$ sufficiently small only depending on ω and s_c-1 such that for $0<r<r_2$, we have

$$\text{dist}\Big(\omega\log(r)+c_2+\alpha_0,\pi\mathbb{Z}\Big)<\delta_1\ \Rightarrow\ r^{\frac12}|Q(r)-\Phi_*(r)|\ge\frac{4}{s_c-1}r^{\frac12}|\Lambda Q(r)|+\frac{c_1\sin(\alpha_0)}{2}.$$

In view of (2.42), we infer for $k\ge k_1$ large enough

$$\text{dist}\left(\omega\log\left(\frac{r_0}{\mu_k}\right)+c_2+\alpha_0,\pi\mathbb{Z}\right)\ge\delta_1 \quad (2.45)$$

and (2.41) is proved.

step 8 Counting. We have so far obtained

$$\begin{aligned}&\#\{r\ge0 \text{ such that } \Lambda u_k(r)=0\}\\ =\ &\#\left\{0\le r\le\frac{r_0}{\mu_k}\text{ such that }\Lambda Q(r)=0\right\}+\#\{r>r_0\text{ such that }\Lambda\psi_1(r)=0\}\end{aligned}$$

which implies

$$\#\{r \geq 0 \text{ such that } \Lambda u_{k+1}(r) = 0\} \quad = \quad \#\{r \geq 0 \text{ such that } \Lambda u_k(r) = 0\} + \#A_k,$$

with

$$A_k := \left\{ \frac{r_0}{\mu_k} < r \leq \frac{r_0}{\mu_{k+1}} \text{ such that } \Lambda Q(r) = 0 \right\}.$$

We claim for $k \geq k_0$ large enough:

$$\#A_k = 1 \tag{2.46}$$

which by possibly shifting the numerotation by a fixed amount ensures that Λu_k vanishes exactly k times.

Upper bound. We first claim

$$\#A_k \leq 1 \tag{2.47}$$

Recall that

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \text{ as } r \to +\infty, \tag{2.48}$$

so that there exists $R \geq 1$ large enough such that

$$\{r \geq R \,/\, \Lambda Q(r) = 0\} = \{r_q, \;\; q \geq q_1\}, \;\omega \log(r_q) + c_8 = q\pi + O\left(\frac{1}{r_q^{s_c - 1}}\right). \tag{2.49}$$

In view of (2.44) and (2.48), we have

$$c_2 + \alpha_0 = c_8$$

and hence, together with (2.45) and (2.49), we infer

$$\inf_{q \geq q_1, k \geq k_1} \left| \log\left(\frac{r_0}{\mu_k}\right) - \log(r_q) \right| \geq \frac{\delta_1}{2\omega}. \tag{2.50}$$

This implies for $k \geq k_1$

$$\begin{aligned} A_k \quad &= \quad \left\{ q \geq q_1 \text{ such that } r_q \in \left(\frac{r_0}{\mu_k}, \frac{r_0}{\mu_{k+1}}\right) \right\} \qquad (2.51) \\ &\subset \quad \left\{ q \geq q_1 \text{ such that } \log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega} \leq \log(r_q) \leq \log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} \right\}. \end{aligned}$$

Since $\lambda_{k,+} < \mu_k < \lambda_{k,-}$ with $\lambda_{k,\pm}$ given by (2.38), we have for $k \geq k_1$

$$\begin{aligned} &\log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right) = \log(\mu_k) - \log(\mu_{k+1}) - \frac{\delta_1}{\omega} \\ \leq \quad &\log(\lambda_{k_+}) - \log(\lambda_{k+1,-}) - \frac{\delta_1}{\omega} \leq \frac{\pi + 2\delta_0 - \delta_1}{\omega}. \end{aligned}$$

Also, we have for $q \geq q_1$

$$\log(r_{q+1}) - \log(r_q) \quad = \quad \frac{\pi}{\omega} + O\left(\frac{1}{r_q^{s_c - 1}}\right).$$

We now choose δ_0 such that

$$0 < \delta_0 < \frac{\delta_1}{4}. \tag{2.52}$$

Then, we infer for $k \geq k_1$

$$\log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right) \leq \frac{\pi}{\omega} - \frac{\delta_1}{2\omega}$$

and hence for $k \geq k_1$ and $q \geq q_1$, we have

$$\log(r_{q+1}) - \log(r_q) \quad > \quad \log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right)$$

which in view of (2.51) implies (2.47).

Lower bound. We now prove (2.46) and assume by contradiction:

$$\# A_{k_2} = 0.$$

Then, let $q_2 \geq q_1$ such that

$$r_{q_2} < \frac{r_0}{\mu_{k_2}} < \frac{r_0}{\mu_{k_2+1}} < r_{q_2+1}.$$

We infer from (2.50):

$$\log(r_{q_2}) \leq \log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega} < \log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} \leq \log(r_{q_2+1}). \tag{2.53}$$

However, we have for $k \geq k_1$

$$\begin{aligned} & \log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega}\right) = \log(\mu_{k_2}) - \log(\mu_{k_2+1}) + \frac{\delta_1}{\omega} \\ \geq \;& \log(\lambda_{k_2,-}) - \log(\lambda_{k_2+1,+}) + \frac{\delta_1}{\omega} \geq \frac{\pi - 2\delta_0 + \delta_1}{\omega} \geq \frac{\pi}{\omega} + \frac{\delta_1}{2\omega} \end{aligned}$$

in view of our choice (2.52). Hence, we infer

$$\log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega}\right) > \log(r_{q_2+1}) - \log(r_{q_2})$$

which contradicts (2.53).
This concludes the proof of Proposition 2.5. $\square$

We now collect final estimates on the constructed solution Φ_n which conclude the proof of Proposition 1.1.

Corollary 2.6. *Let Φ_n the solution to* (2.1) *constructed in Proposition 2.5. Then there exists a small enough constant $r_0 > 0$ independent of n such that:*
1. Convergence to Φ_* as $n \to +\infty$*:*

$$\lim_{n\to+\infty} \sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0. \tag{2.54}$$

2. Convergence to Q at the origin*: there holds for some $\mu_n \to 0$ as $n \to +\infty$:*

$$\lim_{n\to+\infty} \sup_{r \leq r_0} \left| \Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right) \right| = 0. \tag{2.55}$$

3. Last zeroes*: let $r_{0,n} < r_0$ denote the last zero of $\Lambda\Phi_n$ before r_0. Then, for $n \geq N$ large enough, we have*

$$e^{-\frac{2\pi}{\omega}} r_0 < r_{0,n} < r_0.$$

Let $r_{\Lambda Q,n} < r_0/\mu_n$ denote the last zero of ΛQ before r_0/μ_n, then

$$r_{0,n} = \mu_n r_{\Lambda Q,n}(1 + O(r_0^2)).$$

Proof. We choose $r_0 > 0$ small enough as in the proof of Proposition 2.5. We start with the proof of the first claim. Recall from the proof of Proposition 2.5 that we have for $r \geq r_0$

$$\Phi_n(r) = \Phi_*(r) + \varepsilon(\mu_n)\psi_1(r) + \varepsilon(\mu_n)w(r)$$

where we have in particular

$$\sup_{r_0 \leq r \leq 1} r^{\frac{1}{2}}(|\psi_1| + |w|) + \sup_{r \geq 1} r^{\frac{2}{p-1}}(|\psi_1| + |w|) \lesssim 1$$

and

$$\lim_{n \to +\infty} \varepsilon(\mu_n) = 0.$$

We infer

$$\begin{aligned}
&\sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| \\
\lesssim\ & \varepsilon(\mu_n)\left(\sup_{r \geq r_0}(|\psi_1(r)| + |w(r)|) + \sup_{r \geq 1} r^{\frac{2}{p-1}}(|\psi_1(r)| + |w(r)|)\right) \\
\lesssim\ & \varepsilon(\mu_n) r_0^{-\frac{1}{2}}
\end{aligned}$$

and hence

$$\lim_{n \to +\infty} \sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0.$$

Next, recall from the proof of Proposition 2.5 that we have for $r \leq r_0$

$$\Phi_n(r) = \frac{1}{\mu_n^{\frac{2}{p-1}}}(Q + \mu_n^2 T_1)\left(\frac{r}{\mu_n}\right)$$

with

$$\sup_{0 \leq r \leq \frac{r_0}{\mu_n}} (1+r)^{-\frac{3}{2}}|T_1| \lesssim 1.$$

We infer for $r \leq r_0$

$$\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right)\right| \leq \mu_n^{2-\frac{2}{p-1}}|T_1|\left(\frac{r}{\mu_n}\right) \lesssim \mu_n^{\frac{1}{2}-\frac{2}{p-1}}$$

and hence

$$\sup_{r \leq r_0}\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_c - 1}. \tag{2.56}$$

and since $\mu_n \to 0$ as $n \to +\infty$, (2.55) is proved.

We now estimate the localization of the last zeroes of Φ_n and ΛQ before r_0. Recall that

$$\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \text{ as } r \to +\infty.$$

Since $\sin(\omega\log(r) + c_8)$ changes sign on the interval

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n} \leq r \leq \frac{r_0}{\mu_n},$$

and since $r \gg 1$ on this interval, we infer by the mean value theorem that $\Lambda Q(r)$ has a zero on this interval. In particular, this yields

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n} \leq r_{\Lambda Q, n} \leq \frac{r_0}{\mu_n}.$$

Also, recall from the proof of Proposition 2.5 that we have for $r \le r_0$

$$\Lambda\Phi_n(r) = \frac{1}{\mu_n^{\frac{2}{p-1}}}(\Lambda Q + \mu_n^2 \Lambda T_1)\left(\frac{r}{\mu_n}\right),$$

Since

$$\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \text{ as } r \to +\infty,$$

and

$$\sup_{0 \le r \le \frac{r_0}{\mu_n}} (1+r)^{\frac{3}{2}} |\Lambda T_1| \lesssim 1,$$

and since

$$e^{-\frac{2\pi}{\omega}} r_0 \le r \le r_0,$$

we have $r/\mu_n \sim r_0/\mu_n \gg 1$ for $n \ge N$ large enough, we infer

$$\Lambda\Phi_n(r) \sim \frac{c_7 \sin(\omega \log(r) - \omega \log(\mu_n) + c_8) + O(r_0^2)}{\mu_n^{\frac{2}{p-1}}\left(\frac{r}{\mu_n}\right)^{\frac{1}{2}}}.$$

This yields

$$\Big|\omega \log(r_{0,n}) - \omega \log(\mu_n) + c_8 - (\omega \log(r_{\Lambda Q,n}) + c_8)\Big| \lesssim r_0^2$$

and hence

$$\begin{aligned} r_{0,n} &= \mu_n r_{\Lambda Q,n} e^{O(r_0^2)} \\ &= \mu_n r_{\Lambda Q,n}(1 + O(r_0^2)). \end{aligned}$$

Furthermore, since we have

$$e^{-\frac{3\pi}{2\omega}} \frac{r_0}{\mu_n} \le r_{\Lambda Q,n} \le \frac{r_0}{\mu_n},$$

we deduce

$$e^{-\frac{2\pi}{\omega}} r_0 \le r_{0,n} \le r_0.$$

This concludes the proof of the corollary. □

3. Spectral gap in weighted norms

Our aim in this section is to produce a spectral gap for the linearized operator corresponding to (1.4) around Φ_n:

$$\mathcal{L}_n := -\Delta + \Lambda - p\Phi_n^{p-1}. \tag{3.1}$$

Recall (1.18), then $\mathcal{L}_n$ is self adjoint for the L^2_ρ scalar product. Moreover, from (A.1) and the local compactness of the Sobolev embeddings $H^1(|x| \le R) \hookrightarrow L^2(|x| \le R)$, and the fact that $\Phi_n \in L^\infty$, the selfadjoint operator $\mathcal{L}_n + M_n$ for the measure ρdx is for $M_n \ge 1$ large enough invertible with compact resolvent. Hence $\mathcal{L}_n$ is diagonalizable in a Hilbert basis of L^2_ρ, and we claim the following sharp spectral gap estimate:

Proposition 3.1 (Spectral gap for $\mathcal{L}_n$). *Let $n > N$ with $N \gg 1$ large enough, then the following holds:*
1. Eigenvalues. *The spectrum of $\mathcal{L}_n$ is given by*

$$-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2 < -\mu_{-1,n} = -1 < 0 < \lambda_{0,n} < \lambda_{1,n} < \dots \tag{3.2}$$

with

$$\lambda_{j,n} > 0 \text{ for all } j \geq 0 \text{ and } \lim_{j\to+\infty} \lambda_{j,n} = +\infty. \tag{3.3}$$

The eigenvalues $(-\mu_{j,n})_{1\leq j\leq n+1}$ *are simple and associated to spherically symmetric eigenvectors*

$$\psi_{j,n}, \quad \|\psi_{j,n}\|_{L^2_\rho} = 1, \quad \psi_{1,n} = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_\rho},$$

and the eigenspace for $\mu_{-1,n}$ *is spanned by*

$$\psi^k_{-1,n} = \frac{\partial_k \Phi_n}{\|\partial_k\Phi_n\|_\rho}, \quad 1 \leq k \leq 3. \tag{3.4}$$

Moreover, there holds as $r \to +\infty$

$$|\partial_k \psi_{j,n}(r)| \lesssim (1+r)^{-\frac{2}{p-1}-\mu_{j,n}-k}, \quad 1 \leq j \leq n+1, \quad k \geq 0. \tag{3.5}$$

2. Spectral gap. *There holds for some constant* $c_n > 0$*:*

$$\forall \varepsilon \in H^1_\rho, \ (\mathcal{L}_n\varepsilon, \varepsilon)_\rho \geq c_n\|\varepsilon\|^2_{H^1_\rho} - \frac{1}{c_n}\left[\sum_{j=1}^{n+1}(\varepsilon, \psi_{j,n})^2_\rho + \sum_{k=1}^{3}(\varepsilon, \psi^k_{0,n})^2_\rho\right]. \tag{3.6}$$

In other words, $\mathcal{L}_n$ admits $n+1$ instability directions when $\Lambda\Phi_n$ vanishes n times, and 0 is never in the spectrum. Moreover, there are no additional non radial instabilities apart from the trivial translation invariance (3.4).

The rest of this section is devoted to preparing the proof of Proposition 3.1 which is completed in section 3.4.

3.1. **Decomposition in spherical harmonics.** We first recall some basic facts about spherical harmonics. Spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere $\mathbb{S}^2$. The spectrum of this self-adjoint operator with compact resolvent is

$$\{-m(m+1), \ m \in \mathbb{N}\}.$$

For each $m \in \mathbb{N}$ the eigenvalue $m(m+1)$ has geometric multiplicity $2m+1$. We then denote the associated orthonormal family of eigenfunctions by $(Y^{(m,k)})_{m\in\mathbb{N}, \ -m\leq k\leq m}$ so that we have

$$L^2(\mathbb{S}^2) = \bigoplus_{m=0}^{+\infty} \text{Span}\left\langle Y^{(m,k)}, \ -m \leq k \leq m\right\rangle$$

and

$$-\Delta_{\mathbb{S}^2} Y^{(m,k)} = m(m+1)Y^{(m,k)}, \quad \int_{\mathbb{S}^2} Y^{(m,k)} Y^{(m',k')} d\sigma_{\mathbb{S}^2} = \delta_{(m,k),(m',k')}. \tag{3.7}$$

In particular, $u \in H^1_\rho$ is decomposed as

$$u = \sum_{m=0}^{+\infty}\sum_{k=-m}^{m} u_{m,k} Y^{(m,k)}$$

where $u_{m,k}$ are radial functions satisfying the Parseval formula

$$\|u\|^2_\rho = \sum_{m=0}^{+\infty}\sum_{k=-m}^{m} \|u_{m,k}\|^2_\rho.$$

This allows us to write

$$(\mathcal{L}_n(u), u)_\rho = \sum_{m=0}^{+\infty}\sum_{k=-m}^{m} (\mathcal{L}_{n,m}(u_{m,k}), u_{m,k})_\rho \tag{3.8}$$

where we recall

$$\mathcal{L}_{n,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_n^{p-1}.$$

We also recall for further use the definition of the operators:

$$\mathcal{L}_{\infty,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_*^{p-1},$$
$$H_m := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}.$$

3.2. **Linear ODE analysis.** We compute in this section the fundamental solutions of $\mathcal{L}_{n,m}$, H_m and we recall the behavior of the eigenvalues of $\mathcal{L}_\infty$. The claims are standard and follow from a classical ODE perturbation analysis using in an essential way the uniform bound (1.10).

Lemma 3.2 (Fundamental solution for $\mathcal{L}_{n,m}$, H_m). *Let $m \geq 1$. Let $\Delta_m > 0$ be given by* (C.1).
1. Basis for $\mathcal{L}_{n,m}$. *Let $\phi_{n,m}$ be the solution to $\mathcal{L}_{n,m}\phi_{n,m} = 0$ with the behaviour at the origin*

$$\varphi_{n,m} = r^m[1 + O(r^2)] \quad as \quad r \to 0, \tag{3.9}$$

then

$$\varphi_{n,m} \sim \frac{c_1}{r^{\frac{2}{p-1}}} + c_2 r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}} \ as \ r \to +\infty, \quad (c_1, c_2) \neq (0,0). \tag{3.10}$$

2. Basis for H_1: *let $m = 1$, then there exists a fundamental basis (ν_1, ϕ_1) with*

$$\nu_1(r) = \frac{Q'(r)}{Q''(0)} \left| \begin{array}{l} = r[1 + O(r^2)] \quad as \quad r \to 0 \\ \sim \frac{c_{1,+}}{r^{\frac{1+\sqrt{\Delta_1}}{2}}} \quad as \quad r \to +\infty \end{array} \right. \tag{3.11}$$

and

$$\phi_1(r) = \left| \begin{array}{l} \frac{1}{r^2}[1 + O(r^2)] \quad as \quad r \to 0 \\ \sim \frac{c_{1,-}}{r^{\frac{1-\sqrt{\Delta_1}}{2}}} \quad as \quad r \to +\infty, \quad c_{1,-} \neq 0. \end{array} \right. \tag{3.12}$$

2. Basis for H_m: *let $m \geq 2$, then there exists a fundamental basis (ν_m, ϕ_m) with*

$$\nu_m \left| \begin{array}{l} = r^m[1 + O(r^2)] \quad as \quad r \to 0 \\ \sim \frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} \quad as \ r \to +\infty, \quad c_{m,-} > 0 \end{array} \right. \tag{3.13}$$

and

$$\phi_m(r) = \left| \begin{array}{l} \frac{1}{r^{1+m}}[1 + O(r^2)] \quad as \quad r \to 0 \\ \sim \frac{c_{m,+}}{r^{\frac{1+\sqrt{\Delta_m}}{2}}} \quad as \quad r \to +\infty, \quad c_{m,+} \neq 0. \end{array} \right. \tag{3.14}$$

4. Positivity*:*

$$\nu_m(r) > 0 \quad on \quad (0, +\infty). \tag{3.15}$$

5. Uniform closeness*: Fix $m \geq 1$. There exists a sequence*[13] *$\mu_n \to 0$ as $n \to +\infty$ such that for $n \geq N$ large enough*

$$\sup_{0 \le r \le r_0} \frac{\left|\mu_n^{-m}\varphi_{n,m}(r) - \nu_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_m\left(\frac{r}{\mu_n}\right)\right|} + \sup_{0 \le r \le r_0} \frac{\left|\mu_n^{-m+1}\varphi'_{n,m}(r) - \nu'_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu'_m\left(\frac{r}{\mu_n}\right)\right|} \lesssim r_0^2. \tag{3.16}$$

[13] $(\mu_n)_{n \geq N}$ is the same sequence of scales as in (1.10) in Proposition 1.1 and Corollary 2.6.

The uniform in n bound (3.16) follows from the uniform control (1.10) using a standard ODE analysis. We provide a detailed proof of Lemma 3.2 in Appendix C for the sake of completeness.

We now detail the structure of the smooth zero of $\mathcal{L}_{n,0}$ which is the key to the counting of non positive eigenvalues. Let $\varphi_{n,0}$ be the solution to

$$\mathcal{L}_{n,0}(\varphi_{n,0}) = 0, \quad \varphi_{n,0}(0) = 1, \quad \varphi'_{n,0}(0) = 0. \tag{3.17}$$

We recall that $r_{0,n} < r_0$ denotes the last zero of $\Lambda\Phi_n$ before r_0, and we let $r_{1,n} < r_0$ denote the last zero of $\varphi_{n,0}$ before r_0. We claim:

Lemma 3.3 (Zeroes of $\Phi_{n,0}$)**.** *There holds*

$$\sup_{0\le r\le r_0} \left(1+\frac{r}{\mu_n}\right)^{\frac{1}{2}} \left|\varphi_{n,0}(r) - \frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right| \lesssim r_0^2 \tag{3.18}$$

and

$$r_{1,n} = r_{0,n} + O(r_0^3), \quad e^{-\frac{2\pi}{\omega}} r_0 \le r_{1,n} \le r_0. \tag{3.19}$$

This is again a simple perturbative analysis which proof is detailed in Appendix D.

We now claim the following classical result which relies on the standard analysis of explicit special functions:

Lemma 3.4 (Special functions lemma)**.** *Let $\lambda \in \mathbb{R}$. The solutions to*

$$\mathcal{L}_\infty(\psi) = \lambda\psi, \quad \psi \in H^1_\rho(1,+\infty)$$

behaves for $r \to +\infty$ as

$$\psi \sim r^{-\frac{2}{p-1}+\lambda}$$

and for $r \to 0_+$ as

$$\psi = \frac{1}{r^{\frac{1}{2}}}\cos(\omega \log(r) - \Phi(\lambda)) + O\left(r^{\frac{3}{2}}\right) \tag{3.20}$$

where

$$\Phi(\lambda) = \arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)}\right).$$

Proof. We consider the solution ψ to

$$\mathcal{L}_\infty(\psi) = \lambda\psi.$$

The change of variable and unknown

$$\psi(r) = \frac{1}{(2z)^{\frac{\gamma}{2}}} w(z), \quad z = \frac{r^2}{2}$$

leads to

$$\mathcal{L}_\infty(\psi) - \lambda\psi = -\frac{2}{(2z)^{\frac{\gamma}{2}}}\left(zw''(z) + \left(-\gamma + \frac{3}{2} - z\right)w'(z) - \left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2}\right)w(z)\right)$$

and thus $\mathcal{L}_\infty(\psi) = \lambda\psi$ if and only if

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$$

with

$$a = \frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}. \tag{3.21}$$

Hence w is a linear combination of the special functions $M(a,b,z)$ and $U(a,b,z)$ whose asymptotic at infinity is given by (2.9):

$$M(a,b,z)\sim\frac{\Gamma(b)}{\Gamma(a)}z^{a-b}e^{z},\quad U(a,b,z)\sim z^{-a}\text{ as }z\to+\infty,$$

In particular, a non zero contribution of $M(a,b,z)$ to w would yield for $\psi(r)$ the following asymptotic

$$\psi(r)\sim r^{\frac{2}{p-1}-3-\lambda}e^{\frac{r^2}{2}}\text{ as }r\to+\infty.$$

which contradicts $\psi\in H^1_\rho(1,+\infty)$. Hence

$$w(z)=U(a,b,z).$$

In view of the asymptotic of U recalled in (2.9), we have

$$w(z)\sim z^{-a}\text{ as }z\to+\infty.$$

Since

$$\psi(r)=\frac{1}{r^\gamma}w\left(\frac{r^2}{2}\right),$$

this yields

$$\psi\sim r^{-\frac{2}{p-1}+\lambda}\text{ as }r\to+\infty.$$

Also, in view of the asymptotic of U recalled in (2.11), we have

$$w(z)=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+\frac{\Gamma(1-b)}{\Gamma(a-b+1)}+O(z^{2-\Re(b)})\text{ as }z\to 0,$$

which in view of (3.21) and the fact that $\gamma=1/2+i\omega$ yields

$$w(z)=\frac{\Gamma(-i\omega)}{\Gamma\left(\frac{1}{p-1}-\frac{\lambda}{2}-\frac14-\frac{i\omega}{2}\right)}z^{i\omega}+\frac{\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1}-\frac{\lambda}{2}-\frac14+\frac{i\omega}{2}\right)}+O(z)\text{ as }z\to 0.$$

Since

$$\psi(r)=\frac{1}{r^\gamma}w\left(\frac{r^2}{2}\right),$$

this yields

$$\begin{aligned}\psi(r)\quad=\quad&\frac{2^{-\frac{i\omega}{2}}}{r^{\frac12}}\left(\frac{2^{-\frac{i\omega}{2}}\Gamma(-i\omega)}{\Gamma\left(\frac{1}{p-1}-\frac{\lambda}{2}-\frac14-\frac{i\omega}{2}\right)}r^{i\omega}+\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1}-\frac{\lambda}{2}-\frac14+\frac{i\omega}{2}\right)}r^{-i\omega}\right)\\&+O\left(r^{\frac32}\right)\text{ as }r\to 0,\end{aligned}$$

and since ψ is real valued, we infer[14]

$$\psi(r)=\frac{\cos(\omega\log(r)-\Phi(\lambda))}{r^{\frac12}}+O\left(r^{\frac32}\right)\text{ as }r\to 0,$$

where

$$\Phi(\lambda)=\arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1}-\frac{\lambda}{2}-\frac14+\frac{i\omega}{2}\right)}\right).$$

This concludes the proof of the lemma. $\square$

[14]Note in particular that Γ satisfies $\overline{\Gamma(z)}=\Gamma(\overline{z})$ for all $z\in\mathbb{C}$.

3.3. **Perturbative spectral analysis.** We now prove elementary spectral analysis perturbation results based on the uniform bounds (1.9), (1.10) which allow us to precisely count the number of instabilities of $\mathcal{L}_{n,0}$.

Lemma 3.5 (Control of the outside spectrum)**.** *Let $r_0>0$ and let $r_{n,2}$ such that $r_{n,2}>e^{-\frac{2\pi}{\omega}}r_0$. Let us define the operators*

$$\left|\begin{array}{l} A_n[r_{n,2}](f)=\mathcal{L}_{n,0}(f) \ \ on\ r>r_{n,2}, \quad f(r_{n,2})=0,\\ A_\infty[r_{n,2}](f)=\mathcal{L}_{\infty}(f) \ \ on\ r>r_{n,2}, \quad f(r_{n,2})=0,\end{array}\right. \tag{3.22}$$

then

$$\mathop{sup}_{\lambda\in Spec(A_n[r_{n,2}])}\ \mathop{inf}_{\mu\in Spec(A_\infty[r_{n,2}])}|\lambda-\mu|+\mathop{sup}_{\mu\in Spec(A_\infty[r_{n,2}])}\ \mathop{inf}_{\lambda\in Spec(A_n[r_{n,2}])}|\lambda-\mu|\to 0 \tag{3.23}$$

as $n\to+\infty$.

Proof. In view of (A.1), the local compactness of the Sobolev embeddings

$$H^1(|x|\leq R)\hookrightarrow L^2(|x|\leq R) \text{ for all } 1\leq R<+\infty,$$

and the fact that $\Phi_n\in L^\infty$ and , the selfadjoint operators $A_n[r_{n,2}]+M_n$ for the measure ρdx are for $M_n\geq 1$ large enough invertible with compact resolvent, and $A_n[r_{n,2}]$ is diagonalizable. Since $\Phi_*\in L^\infty(r>r_0)$, we deduce similarly that $A_\infty[r_{n,2}]$ is diagonalizable. Let then λ_n be an eigenvalue of $A_n[r_{n,2}]$ with normalized eigenvector w_n:

$$\mathcal{L}_n(w_n)=0 \text{ on } r>r_{n,2}, \quad w_n(r_{n,2})=0, \quad \|w_n\|_{L^2_\rho(r>r_{n,2})}=1.$$

Since $A_\infty[r_{n,2}]$ is diagonalizable in a Hilbert basis of L^2_ρ, we have

$$\begin{array}{rcl}\|A_\infty[r_{n,2}](w_n)-\lambda_n w_n\|_{L^2_\rho(r>r_{n,2})} & \geq & \mathrm{dist}(\lambda_n,\mathrm{spec}(A_\infty[r_{n,2}]))\|w_n\|_{L^2_\rho(r>r_{n,2})}\\ & = & \mathrm{dist}(\lambda_n,\mathrm{spec}(A_\infty[r_{n,2}])).\end{array}$$

On the other hand,

$$\|A_\infty[r_{n,2}](w_n)-\lambda_n w_n\|_{L^2_\rho(r>r_{n,2})} \ = \ \|(A_\infty[r_{n,2}]-A_n[r_{n,2}])(w_n)\|_{L^2_\rho(r>r_{n,2})}$$

from which:

$$\begin{array}{l}\mathrm{dist}(\lambda_n,\mathrm{spec}(A_\infty[r_{n,2}]))\leq\|(A_\infty[r_{n,2}]-A_n[r_{n,2}])(w_n)\|_{L^2_\rho(r>r_{n,2})}\\ \leq \ \left(\sup_{r\geq r_{n,2}}\left(p|\Phi_n(r)-\Phi_*(r)|^{p-1}\right)\right)^{\frac12}\|w_n\|_{L^2_\rho(r>r_{n,2})}\leq\left(\sup_{r\geq r_{n,2}}\left(p|\Phi_n(r)-\Phi_*(r)|^{p-1}\right)\right)^{\frac12}\\ \to \ 0 \ \text{ as } \ n\to+\infty\end{array}$$

from (1.9). (3.23) follows by exchanging the role $A_n[r_{n,2}]$ and $A_\infty[r_{n,2}]$. $\square$

Lemma 3.6 (Local continuity of the spectrum)**.** *Let $r_0>0$ and let r_1 and r_2 such that*

$$e^{-\frac{2\pi}{\omega}}r_0\leq r_1,r_2\leq r_0$$

and

$$r_1=r_2+O(r_0^3).$$

Then, for any eigenvalue λ_1 of $A_\infty[r_1]$ such that $\lambda_1\in[-3,1]$, we have

$$dist(\lambda_1,Spec(A_\infty[r_2]))\lesssim r_0^{\frac32}. \tag{3.24}$$

Proof. Recall from the proof of Lemma 3.5 that both $A_\infty[r_1]$ and $A_\infty[r_2]$ are diagonalizable. Furthermore, by Sturm-Liouville, their eigenvalues are simple. Let λ_1 be an eigenvalue of $A_\infty[r_1]$. We claim the existence of a nearby eigenvalue λ_2 of $A_\infty[r_2]$ using a classical Lyapunov Schmidt procedure.
Let φ_1 the normalized eigenfunction of $A_\infty[r_1]$ associated to λ_1 so that

$$A_\infty[r_1](\varphi_1)=\lambda_1\varphi_1,\quad \|\varphi_1\|_\rho=1.$$

The eigenvalue equation

$$A_\infty[r_2](\varphi_2)=\lambda_2\varphi_2$$

is equivalent to

$$A_\infty[r_1](g)=\lambda_2 g+hg+(r_2-r_1)\partial_r g \tag{3.25}$$

where

$$g(r)=\varphi_2(r+r_2-r_1),\quad h(r)=\frac{pc_\infty^{p-1}}{(r+r_2-r_1)^2}-\frac{pc_\infty^{p-1}}{r^2}.$$

We decompose

$$g=\varphi_1+r_0\tilde{g},\quad \lambda_2=\lambda_1+cr_0$$

where the constant c will be chosen later. Then, g satisfies (3.25) if and only if $\tilde{g}$ satisfies

$$(A_\infty[r_1]-\lambda_1)(\tilde{g})=c\varphi_1+cr_0\tilde{g}+\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r g. \tag{3.26}$$

We choose c such that

$$c(\varphi_1,r_0,\tilde{g}):=-\frac{1}{1+r_0(\tilde{g},\varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g}),\varphi_1\right)_\rho.$$

Then, the right-hand side of (3.26) is orthogonal to φ_1 and hence to the kernel of $A_\infty[r_1]-\lambda_1$ since λ_1 is a simple eigenvalue. Thus, we infer

$$\tilde{g}=\mathcal{F}(\tilde{g}) \tag{3.27}$$

where

$$\mathcal{F}(\tilde{g}):=B_\infty[r_1,\lambda_1]^{-1}\Big(c(\varphi_1,r_0,\tilde{g})(\varphi_1+r_0\tilde{g})+\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g})\Big)$$

with the operator $B_\infty[r_1,\lambda_1]$ being the restriction of $A_\infty[r_1]-\lambda_1$ to the orthogonal complement of the kernel of $A_\infty[r_1]-\lambda_1$, i.e.

$$B_\infty[r_1,\lambda_1]=(A_\infty[r_1]-\lambda_1)_{|_{\varphi_1^\perp}}.$$

Since λ_1 is an eigenvalue of $A_\infty[r_1]$, from the explicit behavior (3.20) of the eigenfunctions of $\mathcal{L}_\infty$ and the boundary condition (3.22) at r_1 one deduces that there exists $k\in\mathbb{Z}$ such that

$$\omega\log(r_1)-\Phi(\lambda_1)=k\pi+\frac{\pi}{2}+O(r_0^2).$$

Let λ_1' be the smallest eigenvalue of $A_\infty[r_1]$ greater than λ_1. It then satisfies:

$$\omega\log(r_1)-\Phi(\lambda_1')=k\pi+\frac{\pi}{2}\pm\pi+O(r_0^2)$$

and so

$$|\Phi(\lambda_1)-\Phi(\lambda_1')|=\pi+O(r_0^2)\geq\frac{\pi}{2}.$$

As Φ is a continuous function we deduce that there exists $c>0$ independent of r_0 such that $\lambda_1'\geq\lambda_1+c$ and we infer

$$\inf\{|\lambda-\lambda_1|,\ \lambda\in\mathrm{Spec}(A_\infty[r_1]),\ \lambda>\lambda_1\}\geq c.$$

Similarly
$$\inf\{|\lambda-\lambda_1|,\ \lambda\in \mathrm{Spec}(A_\infty[r_1]),\ \lambda<\lambda_1\}\geq c',\quad c'>0$$
and we conclude that
$$\|B_\infty[r_1,\lambda_1]^{-1}\|_{\mathcal{L}(L^2_\rho,H^2_\rho)}\lesssim 1$$
with a bound that does not depend on r_0. Also, note that
$$\frac{h}{r_0} \;=\; \frac{pc_\infty^{p-1}}{r_0r^2}\left(\frac{1}{(1+\frac{r_2-r_1}{r})^2}-1\right)=\frac{h_1(r)}{r^2}$$
where
$$h_1(r) \;=\; -pc_\infty^{p-1}\frac{\left(\frac{2(r_1-r_2)}{r_0r}+\frac{(r_1-r_2)^2}{r_0r^2}\right)}{(1+\frac{r_2-r_1}{r})^2}.$$
Since
$$e^{-\frac{2\pi}{\omega}}r_0\leq r_1\leq r_0 \text{ and } r_1=r_2+O(r_0^3),$$
we infer
$$\|h_1\|_{L^\infty(r>r_1)} \;\lesssim\; \|h_1\|_{L^\infty(r>e^{-\frac{2\pi}{\omega}}r_0)}\lesssim r_0.$$
Moreover,
$$\|r^{-2}\|_{L^2(r_1<r<1)} \;\lesssim\; \left(\int_{r_1}^1\frac{dr}{r^2}\right)^{\frac12}\lesssim\frac{1}{r_1^{\frac12}}\lesssim\frac{1}{r_0^{\frac12}}.$$
Collecting the previous estimates, we infer
$$\begin{aligned}
&\|\mathcal{F}(\tilde g)\|_{H^2_\rho(r>r_1)}\\
\lesssim\;&\|B_\infty[r_1,\lambda_1]^{-1}\|_{\mathcal{L}(L^2_\rho,H^2_\rho)}\left\|c(\varphi_1,r_0,\tilde g)(\varphi_1+r_0\tilde g)+\frac{h}{r_0}(\varphi_1+r_0\tilde g)+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde g)\right\|_{L^2_\rho}\\
\lesssim\;&|c(\varphi_1,r_0,\tilde g)|(1+r_0\|\tilde g\|_{L^2_\rho})+r_0\|\tilde g\|_{H^1_\rho}\\
&+\|h_1\|_{L^\infty(r>r_1)}(1+r_0\|\tilde g\|_{L^2_\rho}+\|\varphi_1+r_0\tilde g\|_{L^\infty_{(r_1<r<1)}}\|r^{-2}\|_{L^2(r_1<r<1)})\\
\lesssim\;&\frac{r_0^{\frac12}}{1-r_0\|\tilde g\|_{L^2_\rho}}(1+r_0\|\tilde g\|_{L^2_\rho})+r_0\|\tilde g\|_{H^1_\rho}
\end{aligned}$$
and
$$\|\mathcal{F}(\tilde g_1)-\mathcal{F}(\tilde g_2)\|_{H^2_\rho(r>r_1)} \;\lesssim\; \frac{r_0^{\frac32}}{1-r_0\|\tilde g\|_{L^2_\rho}}(1+r_0\|\tilde g\|_{L^2_\rho})\|\tilde g_1-\tilde g_2\|_{L^2_\rho}+r_0\|\tilde g_1-\tilde g_2\|_{H^1_\rho}.$$
Thus, for $r_0>0$ small enough, the Banach fixed point theorem applies in the space $H^2_\rho(r>r_1)$ and yields a unique solution $\tilde g$ to (3.27) with
$$\|\tilde g\|_{H^2_\rho(r>r_1)}\lesssim r_0^{\frac12}.$$
Hence, φ_2 with
$$\varphi_2(r)=g(r+r_1-r_2),\quad g=\varphi_1+r_0\tilde g$$
satisfies
$$A_\infty[r_2](\varphi_2)=\lambda_2\varphi_2$$

where

$$\begin{aligned}\lambda_2 &= \lambda_1 + c(\varphi_1, r_0, \tilde{g})r_0\\ &= \lambda_1 - \frac{r_0}{1+r_0(\tilde{g},\varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1+r_0\tilde{g}) + \frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g}),\varphi_1\right)_\rho.\end{aligned}$$

Thus, λ_2 belongs to the spectrum of $A_\infty[r_2]$ and hence

$$\begin{aligned}\mathrm{dist}(\lambda_1,\mathrm{Spec}(A_\infty[r_2])) &\leq |\lambda_2-\lambda_1|\\ &\leq \left|\frac{r_0}{1+r_0(\tilde{g},\varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1+r_0\tilde{g}) + \frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g}),\varphi_1\right)_\rho\right|.\end{aligned}$$

In view of the previous estimates, we infer

$$\mathrm{dist}(\lambda_1,\mathrm{Spec}(A_\infty[r_2])) \lesssim \frac{r_0^{\frac32}}{1-r_0\|\tilde{g}\|_{L^2_\rho}}(1+r_0\|\tilde{g}\|_{L^2_\rho}) \lesssim r_0^{\frac32}.$$

and (3.24) is proved. □

3.4. **Proof of Proposition 3.1.** Recall that $\mathcal{L}_n$ is diagonalizable in a Hilbertian basis of L^2_ρ, and hence the spectral gap estimate (3.6) follows from the explicit distribution of eigenvalues (3.2) which we now prove. Observe that the symmetry group of dilations and translations generates the explicit eigenmodes

$$\mathcal{L}_n\Lambda\Phi_n = -2\Lambda\Phi_n, \quad \mathcal{L}_n\nabla\Phi_n = -\nabla\Phi_n. \tag{3.28}$$

Using the decomposition into spherical harmonics (3.8), the further study of the quadratic form $(\mathcal{L}_n(u),u)_\rho$ reduces to the study of the quadratic form $(\mathcal{L}_{n,m}(u),u)_\rho$ for $m\geq 0$ for which classical Strum Liouville arguments are now at hand.

step 1 The case $m=1$. Let $\varphi_{n,1}$ be defined in Lemma 3.2. In particular, $\varphi_{n,1}$ satisfies

$$\mathcal{L}_{n,1}(\varphi_{n,1})=0, \quad \varphi_{n,1}(0)=0, \quad \varphi'_{n,1}(0)=1.$$

Then from standard Sturm Liouville oscillation argument for central potentials, [49], the number of zeros of $\varphi_{n,1}$ in $r>0$ correspond to the number of strictly negative eigenvalues of $\mathcal{L}_{n,1}$.

Since we have

$$\nabla\Phi_n(x) = \Phi_n'(r)\frac{x}{r} = \Phi_n'(r)(Y^{(1,-1)},Y^{(1,1)},Y^{(1,0)})$$

and hence

$$\mathcal{L}_n(\nabla\Phi_n) = -\nabla\Phi_n \text{ implies } \mathcal{L}_{n,1}(\Phi_n') = -\Phi_n'.$$

Thus, $\mathcal{L}_{n,1}$ has at least one strictly negative eigenvalue, and hence $\varphi_{n,1}$ has at least one zero which we denote by $r_{n,1}>0$. On $[0,r_0]$, we have by (3.16):

$$\sup_{0\leq r\leq r_0}\frac{\left|\mu_n^{-1}\varphi_{n,1}(r)-\nu_1\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_1\left(\frac{r}{\mu_n}\right)\right|} \lesssim r_0^2$$

Since $\nu_1(r)>0$ for all $r>0$, we infer that $\varphi_{n,1}$ can not vanish on $[0,r_0]$. Hence, $r_{n,1}\geq r_0$.

No other zero. Assume by contradiction that there exists a second zero $r_{n,2} > r_{n,1}$. Let $f_{n,1}$ being given as

$$f_{n,1} := \begin{cases} \varphi_{n,1} & \text{on } r_{n,1} < r < r_{n,2}, \\ 0 & \text{on } r < r_{n,1}, \\ 0 & \text{on } r > r_{n,2}. \end{cases}$$

Then, we have $f_{n,1} \in H^1_\rho$ and

$$(\mathcal{L}_{n,1}(f_{n,1}), f_{n,1})_\rho = 0. \tag{3.29}$$

On the other hand, using (1.5):

$$\begin{aligned} (\mathcal{L}_{\infty,1}(u), u)_\rho &= \|u'\|^2_\rho + \int_0^{+\infty} \frac{2 - pc_\infty^{p-1}}{r^2} u^2 r^2 \rho dr \\ &= \|u'\|^2_\rho + \frac{2(p+1)}{(p-1)^2} \left(\int_0^{+\infty} \frac{u^2}{r^2} r^2 \rho dr \right) \gtrsim \|\frac{u}{r}\|^2_{L^2_\rho}. \end{aligned} \tag{3.30}$$

We now estimate from (1.10)

$$\sup_{r \geq r_0} r^2 |\Phi_n^{p-1} - (\Phi_*)^{p-1}| = o_{n \to +\infty}(1) \tag{3.31}$$

and hence for u supported in $(r_0, +\infty)$:

$$\begin{aligned} |(\mathcal{L}_{\infty,1}(u), u)_\rho - (\mathcal{L}_{n,1}(u), u)_\rho| &\lesssim \int_{r_0}^{+\infty} \left| \Phi_n^{p-1} - \Phi_*^{p-1} \right| u^2 r^2 \rho(r) dr \\ &\leq o_{n \to +\infty}(1) \left\| \frac{u}{r} \right\|^2_{L^2_\rho}. \end{aligned} \tag{3.32}$$

Since $f_{n,1}$ is supported in $(r_{n,1}, r_{n,2}) \subset (r_0, +\infty)$, (3.30), (3.32) applied to $f_{n,1}$ and (3.29) yield a contradiction for $n \geq N$ large enough. Thus, $r_{n,2}$ can not exist, and hence $\varphi_{n,1}$ vanishes only once.

$\varphi_{n,1}$ *is not an eigenstate.* Since $\varphi_{n,1}$ vanishes only once, $\mathcal{L}_{n,1}$ has exactly one strictly negative eigenvalue. It remains to check the $\varphi_{n,1} \notin L^2_\rho$, i.e. $\varphi_{n,1}$ is not an eigenvector associated to the eigenvalue 0. To this end, note that $\varphi_{n,1}$ is strictly positive on $(0, r_{n,1})$ from (3.9) and strictly negative on $(r_{n,1}, +\infty)$. In particular, we have

$$\varphi'_{n,1}(r_{n,1}) < 0.$$

Since $\mathcal{L}_{n,1}(\varphi_{n,1}) = 0$, we have

$$(r^2 \rho \varphi'_{n,1})' = r^2 \rho \left[\frac{2}{p-1} + \frac{(2 - pr^2 \Phi_n^{p-1})}{r^2} \right] \varphi_{n,1}$$

and from (3.32) for $r \geq r_{n,1} \geq r_0$:

$$2 - r^2 p \Phi_n^{p-1} = 2 - pc_\infty^{p-1} + pc_\infty^{p-1} - r^2 p \Phi_n^{p-1} \geq \frac{2(p+1)}{(p-1)^2} + o(1) > 0. \tag{3.33}$$

Since $\varphi_{n,1}$ is strictly negative on $(r_{n,1}, +\infty)$, we deduce

$$r^2 \rho \varphi'_{n,1}(r) \leq r_{n,1}^2 \rho(r_{n,1}) \varphi'_{n,1}(r_{n,1}) = c_1 < 0 \text{ on } (r_{n,1}, +\infty)$$

which implies

$$\int_{r_{n,1}}^{+\infty} |\varphi'_{n,1}(r)|^2 \rho r^2 dr \gtrsim \int_{r_{n,1}}^{+\infty} \frac{dr}{r^2 \rho} = +\infty$$

and hence $\varphi_{n,1} \notin H^1_\rho$ and is therefore not an eigenvector.

Conclusion. We conclude that -1 is the only negative eigenvalue of $\mathcal{L}_{n,1}$, and is associated to the single eigenvector Φ_n'. Hence, there exists a constant $c_n>0$ such that for all $u\in H^1_\rho$:

$$(\mathcal{L}_{n,1}(u),u)_\rho \geq c_n\|u\|^2_{L^2_\rho} - \frac{1}{c_n}(u,\Phi_n')^2_\rho. \tag{3.34}$$

step 2 The case $m\geq 2$. Let $\varphi_{n,m}$ be defined in Lemma 3.2. In particular, $\varphi_{n,m}$ satisfies

$$\mathcal{L}_{n,m}(\varphi_{n,m})=0 \text{ and } \varphi_{n,m}=r^m(1+O(r^2)) \text{ as } r\to 0_+.$$

Then, the number of zeros of $\varphi_{n,m}$ in $r>0$ corresponds to the number of strictly negative eigenvalues of $\mathcal{L}_{n,m}$. On $[0,r_0]$, we have by Lemma 3.2.

$$\sup_{0\leq r\leq r_0}\frac{\left|\mu_n^{-m}\varphi_{n,m}(r)-\nu_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_m\left(\frac{r}{\mu_n}\right)\right|} \lesssim r_0^2$$

and $\nu_m(r)>0$ for all $r>0$, and hence $\varphi_{n,m}$ cannot vanish on $[0,r_0]$:

$$\varphi_{n,m}(r)>0 \text{ on } [0,r_0].$$

Next, we investigate the sign of $\varphi_{n,m}'(r_0)$. Recall (3.13):

$$\nu_m(r)\sim\frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} \text{ as } r\to+\infty \quad c_{m.-}>0$$

and hence

$$\nu_m'(r)\sim\frac{c_{m,-}(\sqrt{\Delta_m}-1)}{r^{\frac{3-\sqrt{\Delta_m}}{2}}} \text{ as } r\to+\infty.$$

We infer for $n\geq N$ large enough

$$\varphi_{n,m}(r_0)=\frac{c_{m,-}(1+O(r_0^2))\mu_n^m}{\left(\frac{r_0}{\mu_n}\right)^{\frac{1-\sqrt{\Delta_m}}{2}}}$$

and

$$\varphi_{n,m}'(r_0)=\frac{c_{m,-}(\sqrt{\Delta_m}-1)(1+O(r_0^2))\mu_n^{m-1}}{\left(\frac{r_0}{\mu_n}\right)^{\frac{3-\sqrt{\Delta_m}}{2}}}.$$

Thus, taking also into account that $\varphi_{n,m}(r)>0$ on $[0,r_0]$, we infer from the identity for $\varphi_{n,m}(r_0)$ that

$$c_{m,-}>0.$$

Since $\sqrt{\Delta_m}\geq\sqrt{\Delta_1}=\frac{p+3}{p-1}>1$, we conclude:

$$\phi_{n,m}(r_0)>0, \quad \phi_{n,m}'(r_0)>0. \tag{3.35}$$

Since $\mathcal{L}_{n,m}(\varphi_{n,m})=0$, we have

$$(r^2\rho\varphi_{n,m}')'=r^2\rho\left[\frac{2}{p-1}+\frac{(m(m+1)-pr^2\Phi_n^{p-1})}{r^2}\right]\varphi_{n,m} \tag{3.36}$$

which together with (3.35), (3.33) and the fact that $m\geq 2$, and an elementary continuity argument ensures

$$\phi_{m,n}'(r)>0, \quad \phi_{n,m}(r)\geq\phi_{n,m}(r_0)>0 \text{ for } r\geq r_0.$$

Hence $\phi_{n,m}$ does not vanish on $(0,+\infty)$ and using (3.36):

$$r^2\phi_{n,m}'\rho(r)\geq r_0^2\phi_{n,m}'\rho(r_0)=c_0>0$$

which implies
$$\int_{r_0}^{+\infty} (\phi'_{n,m})^2 \rho r^2 dr \gtrsim \int_{r_0}^{+\infty} \frac{dr}{r^2\rho} = +\infty$$
and hence $\phi_{n,m}$ is not eigenvector. We finally conclude that for $m = 2$ and all $n \geq N$ large enough, $\mathcal{L}_{n,2}$ has a spectral gap and there exists a constant $c_n > 0$ such that we have for all $u \in H^1_\rho$
$$(\mathcal{L}_{n,2}(u), u)_\rho \geq c_n \|u\|^2_{L^2_\rho}.$$
Since we have for all $m \geq 2$
$$(\mathcal{L}_{n,m}(u), u)_\rho \geq (\mathcal{L}_{n,2}(u), u)_\rho,$$
we infer for all $m \geq 2$ and for all $u \in H^1_\rho$
$$(\mathcal{L}_{n,m}(u), u)_\rho \geq c_n \|u\|^2_{L^2_\rho}. \tag{3.37}$$

step 3. The case $m = 0$. We now focus onto $\mathcal{L}_{n,0}$ which is the most delicate case, and we claim that $\mathcal{L}_{n,0}$ has exactly $n + 1$ strictly negative eigenvalues, and that 0 is not in the spectrum. The key is to combine the uniform bounds (1.9) with the explicit knowledge of the limiting outer spectrum, Lemma 3.4, as nicely suggested at the formal level in [3].
Let $\varphi_{n,0}$ be the solution to (3.17) so that the number of strictly negative eigenvalues of $\mathcal{L}_{n,0}$ coincides with the numbers of zeroes of $\varphi_{n,0}$. We count the number of zeros of $\varphi_{n,0}$ by comparing them with the number of zeros of $\Lambda\Phi_n$.

Lower bound. First, since $\Lambda\Phi_n$ is an eigenvector of $\mathcal{L}_{n,0}$ corresponding to the eigenvalue -2 and since $\Lambda\Phi_n$ vanishes n times from Proposition 2.5, we infer from Sturm Liouville
$$\#\mathrm{Spec}(\mathcal{L}_{n,0} + 2) \cap (-\infty, 0] = n + 1.$$
In particular, since the number of strictly negative eigenvalues of $\mathcal{L}_{n,0}$ coincides with the number of zeroes of $\varphi_{n,0}$, we infer
$$\#\{r \geq 0 \text{ such that } \varphi_{n,0}(r) = 0\} \geq n + 1.$$
Upper bound. Recall (3.18):
$$\sup_{0 \leq r \leq r_0} \left(1 + \frac{r}{\mu_n}\right)^{\frac{1}{2}} \left|\varphi_{n,0}(r) - \frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right| \lesssim r_0^2.$$
Also, we have $\Lambda Q(0) \neq 0$ and from (2.41):
$$\left(\frac{r_0}{\mu_n}\right)^{\frac{1}{2}} \left|\Lambda Q\left(\frac{r_0}{\mu_n}\right)\right| \geq c > 0$$
for some constant $c > 0$ independent of n. Hence $\varphi_{n,0}$ and ΛQ vanish the same number of times on $[0, r_0]$. Since on the other hand ΛQ and $\Lambda\Phi_n$ vanish the same number of times on $[0, r_0]$ from (2.40), $\varphi_{n,0}$ and $\Lambda\Phi_n$ vanish the same number of times of $[0, r_0]$.
Let now $r_{n,0}$ to be the last zero of $\Lambda\Phi_n$ before r_0. In view of Corollary 2.6, we have
$$e^{-\frac{2\pi}{\omega}} r_0 \leq r_{n,0} \leq r_0.$$
Let us now consider the operators (3.22):
$$A_n[r_{n,0}](f) = \mathcal{L}_{n,0}(f) \text{ on } r > r_{0,n}, \quad f(r_{n,0}) = 0,$$
$$A_\infty[r_{n,0}](f) = \mathcal{L}_\infty(f) \text{ on } r > r_{n,0}, \quad f(r_{n,0}) = 0,$$

then

$$\mathcal{L}_{n,0}(\Lambda\Phi_n) = -2\Lambda\Phi_n \text{ and } \Lambda\Phi_n(r_{n,0}) = 0,$$

implies

$$A_n[r_{n,0}](\Lambda\Phi_n) = -2\Lambda\Phi_n.$$

In particular, -2 belongs to the spectrum of $A_n[r_{n,0}]$. In view of Lemma 3.5, we deduce for $n \geq N$ large enough that the exists an eigenvalue λ_0 of $A_\infty[r_{n,0}]$ such that $\lambda_0 = -2 + o(1)$. On the other hand, in view of Lemma 3.4, the solutions to

$$\mathcal{L}_\infty(f) = \lambda f$$

with $f \in H^1_\rho$ are completely explicit and behave for $r \to 0$ as

$$f \sim \frac{1}{r^{\frac{1}{2}}}\cos(\omega\log(r) - \Phi(\lambda))$$

with

$$\Phi(\lambda) = \arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)}\right).$$

In order for f to be an eigenfunction of $A_\infty[r_{n,0}]$, we need $f(r_{n,0}) = 0$ and hence there should exists $k \in \mathbb{Z}$ such that

$$\omega\log(r_{n,0}) - \Phi(\lambda) \sim \frac{\pi}{2} + k\pi.$$

Recall that $\lambda_0 = -2 + o(1)$ is an eigenvalue of $A_\infty[r_{n,0}]$, and let $\lambda_1 > \lambda_0$ be the next eigenvalue of $A_\infty[r_{n,0}]$. Then, there exists $k_0 \in \mathbb{R}$ such that

$$\omega\log(r_{n,0}) - \Phi(\lambda_0) \sim \frac{\pi}{2} + k_0\pi, \quad \omega\log(r_{n,0}) - \Phi(\lambda_1) \sim \frac{\pi}{2} + (k_0 - 1)\pi$$

and hence

$$\Phi(\lambda_1) = \Phi(-2) + \pi + o(1). \tag{3.38}$$

Now, by numerical check, we have[15]

$$\sup_{5 \leq p < +\infty} \sup_{-2 \leq \lambda \leq 0.5} (\Phi(\lambda) - \Phi(-2) - \pi) \sim -0.5945 < 0,$$

and hence, the solution λ_1 to (3.38) satisfies

$$\inf_{5 \leq p < +\infty} \lambda_1 \geq 0.5 > 0.$$

We infer that $A_\infty[r_{n,0}]$ has no eigenvalue between $\lambda_0 = -2 + o(1)$ and $\lambda_1 \geq 0.5$. Hence, using again Lemma 3.5, $A_n[r_{n,0}]$ has no eigenvalue between -2 and $\lambda_1 + o(1) \geq 0.25$. Thus, we have

$$\#\mathrm{Spec}(A_n[r_{n,0}]) \cap (-\infty, 0] = \#\mathrm{Spec}(A_n[r_{n,0}] + 2) \cap (-\infty, 0].$$

On the other hand, we have

$$\#\mathrm{Spec}(A_n[r_{n,0}] + 2) \cap (-\infty, 0] = \#\{r > r_{n,0} \text{ such that } \Lambda\Phi_n(r) = 0\} + 1$$

[15]Notice that $\Phi(\lambda)$ has a well defined limit as $p \to +\infty$ given by

$$\Phi_\infty(\lambda) = \arg\left(\frac{2^{\frac{i}{4}}\Gamma(\frac{i}{2})}{\Gamma\left(-\frac{\lambda}{2} - \frac{1}{4} + \frac{i}{4}\right)}\right).$$

Our numerics are carried out using Matlab and indicate that $\Phi_p(\lambda)$ is increasing on $[-2, 0.5]$ for all $p \geq 5$ so that the maximum on $[-2, 0.5]$ is achieved at $\lambda = 0.5$. Also, this maximum appears to be a growing function of p so that the maximum in p is given by $\Phi_\infty(0.5) - \Phi_\infty(-2) - \pi \sim -0.5945$. See [3] for a similar numerical computation.

since $\Lambda\Phi_n$ is in the kernel of $A_n[r_{n,0}]+2$, and hence

$$\#\mathrm{Spec}(A_n)\cap(-\infty,0]=\#\{r>r_{n,0}\text{ such that }\Lambda\Phi_n(r)=0\}+1.$$

Also, since $\varphi_{n,0}$ can not be an eigenvector of A_n[16], we have

$$\#\mathrm{Spec}(A_n[r_{n,0}])\cap(-\infty,0]=\#\{r>r_{n,0}\text{ such that }\varphi_{n,0}(r)=0\}.$$

We infer

$$\#\{r>r_{n,0}\text{ such that }\varphi_{n,0}(r)=0\}=\#\{r>r_{n,0}\text{ such that }\Lambda\Phi_n(r)=0\}+1.$$

But since $r_{n,0}$ has been chosen to be the last zero of $\Lambda\Phi_n$ before r_0, we have

$$\#\{r>r_{n,0}\text{ such that }\Lambda\Phi_n(r)=0\}=\#\{r>r_0\text{ such that }\Lambda\Phi_n(r)=0\}$$

and hence

$$\#\{r>r_{n,0}\text{ such that }\varphi_{n,0}(r)=0\}=\#\{r>r_0\text{ such that }\Lambda\Phi_n(r)=0\}+1.$$

Next, together with the fact that $\varphi_{n,0}$ and $\Lambda\Phi_n$ vanish the same number of times of $[0,r_0]$, we infer

$$\begin{aligned}
&\#\{r>0\text{ such that }\varphi_{n,0}(r)=0\}\\
\le\ &\#\{0\le r\le r_0\text{ such that }\varphi_{n,0}(r)=0\}+\#\{r>r_{n,0}\text{ such that }\varphi_{n,0}(r)=0\}\\
=\ &\#\{0\le r\le r_0\text{ such that }\Lambda\Phi_n(r)=0\}+\#\{r>r_0\text{ such that }\Lambda\Phi_n(r)=0\}+1\\
=\ &\#\{r>0\text{ such that }\Lambda\Phi_n(r)=0\}+1\\
=\ &n+1
\end{aligned}$$

and since

$$\#\{r\ge0\text{ such that }\varphi_{n,0}(r)=0\}\ge n+1.$$

$\phi_{n,0}$ *is not an eigenstate*. We conclude that

$$\#\{r\ge0\text{ such that }\varphi_{n,0}(r)=0\}=n+1.$$

Assume now by contradiction that $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$. Recall that $r_{0,n}<r_0$ is the last 0 of $\Lambda\Phi_n$ and let $r_{1,n}<r_0$ be the last 0 of $\varphi_{n,0}$. In particular, we have from Lemma 3.3:

$$e^{-\frac{2\pi}{\omega}}r_0\le r_{0,n},r_{1,n}\le r_0\text{ and }r_{1,n}=r_{0,n}+O(r_0^3).$$

Also, since $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$ and $\varphi_{n,0}(r_{1,n})=0$, we infer that 0 is in the spectrum of $A_n[r_{1,n}]$, and hence applying Lemma 3.5 twice as well as Lemma 3.6, we obtain that

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]),0)\lesssim r_0^{\frac{3}{2}}+o(1)$$

as $n\to+\infty$. In particular, we have for $r_0>0$ small enough and $n\ge N$ large enough

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]),0)\le0.2.$$

On the other hand, we have proved above that $A_n[r_{n,0}]$ has no eigenvalue between -2 and $\lambda_1+o(1)\ge0.25$ so that

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]),0)\ge0.25$$

which is a contradiction. Hence $\varphi_{0,n}$ is not in the kernel of $\mathcal{L}_{n,0}$.

[16] Indeed, $\varphi_{n,0}$ would be an eigenvector for the eigenvalue 0, but 0 is not in the spectrum of A_n as seen above.

Conclusion. We conclude that $\mathcal{L}_{n,0}$ has exactly $n+1$ strictly negative eigenvalues. On the other hand, since $\Lambda\Phi_n$ is an eigenvector of $\mathcal{L}_{n,0}$ corresponding to the eigenvalue -2 and since $\Lambda\Phi_n$ vanishes n times, we infer

$$\#\mathrm{Spec}(\mathcal{L}_{n,0}+2)\cap(-\infty,0]=n+1,$$

and hence $\mathcal{L}_{n,0}$ has exactly $n+1$ negative eigenvalues and the largest negative eigenvalue is -2. We denote these eigenvalues by

$$-\mu_{n+1,n}<\cdots<-\mu_{2,n}<-\mu_{1,n}=-2.$$

By Sturm Liouville, these eigenvalues are simple and associated to eigenvectors

$$\psi_{j,n},\quad \|\psi_{j,n}\|_{L^2_\rho}=1,\quad \psi_{1,n}=\frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_\rho}.$$

Also, there holds for some constant $c_n>0$ and for all $u\in H^1_\rho$

$$(\mathcal{L}_{n,0}(u),u)_\rho\geq c_n\|u\|^2_{L^2_\rho}-\frac{1}{c_n}\left[\sum_{j=1}^{n+1}(u,\psi_{j,n})^2_\rho\right]. \tag{3.39}$$

The behavior as $r\to+\infty$ of the eigenstates (3.5) follows from the asymptotic in Lemma 3.4 and a standard ODE argument using the variation of constants formula, this is left to the reader.

step 4 Conclusion. We decompose $u\in H^1_\rho$ as

$$u=\sum_{m=0}^{+\infty}\sum_{k=-m}^{m}u_{m,k}Y^{(m,k)}$$

where $u_{m,k}$ are radial functions satisfying

$$\|u\|^2_\rho=\sum_{m=0}^{+\infty}\sum_{k=-m}^{m}\|u_{m,k}\|^2_\rho.$$

We have

$$(\mathcal{L}_n(u),u)_\rho = \sum_{m=0}^{+\infty}\sum_{k=-m}^{m}(\mathcal{L}_{n,m}(u_{m,k}),u_{m,k})_\rho.$$

Together with (3.34), (3.37) and (3.39), we infer for all $u\in H^1_\rho$

$$\begin{aligned}(\mathcal{L}_n(u),u)_\rho &= (\mathcal{L}_{n,0}(u_{0,0}),u_{0,0})_\rho+\sum_{k=-1}^{1}(\mathcal{L}_{n,1}(u_{1,k}),u_{1,k})_\rho+\sum_{m=2}^{+\infty}\sum_{k=-m}^{m}(\mathcal{L}_{n,m}(u_{m,k}),u_{m,k})_\rho\\ &\geq c_n\|u\|^2_\rho-\frac{1}{c_n}\left[\sum_{j=1}^{n+1}(u_{0,0},\psi_{j,n})^2_\rho+\sum_{k=1}^{3}(u_{1,k},\Phi_n')^2_\rho\right].\end{aligned}$$

Since $\psi_{j,n}$ are all radial, we have

$$(u_{0,0},\psi_{j,n})_\rho=(u,\psi_{j,n})_\rho.$$

Also, since

$$\nabla\Phi_n(x)=\Phi_n'(r)\frac{x}{r}=\Phi_n'(r)(Y^{(1,-1)},Y^{(1,1)},Y^{(1,0)}),$$

we infer

$$\sum_{k=1}^{3}(u_{1,k},\Phi_n')^2_\rho = \sum_{k=1}^{3}(u,\partial_k\Phi_n)^2_\rho.$$

Finally, there holds for some constant $c_n > 0$ and for all $u \in H^1_\rho$

$$(\mathcal{L}_n u, u)_\rho \geq c_n \|u\|^2_{H^1_\rho} - \frac{1}{c_n}\left[\sum_{j=0}^{n}(u,\psi_{j,n})^2_\rho + \sum_{k=1}^{3}(u,\partial_k\Phi_n)^2_\rho\right].$$

This concludes the proof of Proposition 3.1.

4. Dynamical control of the flow

We now turn to the question of the stability of the self similar solution, and more precisely the construction of a manifold of *finite energy* initial data such that the corresponding solution to (1.1) blows up in finite time with Φ_n profile in the self similar regime described by Theorem 1.2. n is now fixed.

4.1. **Setting of the bootstrap.** We set up in this section the bootstrap analysis of the flow for a suitable set of finite energy initial data. The solution will be decomposed in a suitable way with standard technique, see [32, 38].

Geometrical decomposition. We start by showing the existence of the suitable decomposition. Recall the spectral Proposition 3.1. To ease notations we now omit the n subscript and write ψ_j, μ_j and λ_j instead.

Define the L^∞ tube around the renormalized versions of Φ_n:

$$X_\delta = \left\{ u = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Phi_n + v)\left(\frac{x-y}{\lambda}\right), \ y \in \mathbb{R}^d, \ \lambda > 0, \ \|v\|_{L^\infty} < \delta \right\}$$

Lemma 4.1 (Geometrical decomposition). *There exists $\delta > 0$ and $C > 0$ such that any $u \in X_\delta$ has a unique decomposition*

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Phi_n + \sum_{j=2}^{n+1} a_j\psi_j + \varepsilon)\left(\frac{x-\overline{x}}{\lambda}\right),$$

where ε satisfies the orthogonality conditions

$$(\varepsilon,\psi_j)_\rho = (\varepsilon,\partial_k\Phi_n)_\rho = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3,$$

the parameters λ, $\overline{x}$ and a_j being Fréchet differentiable on X_δ, and with

$$\|\varepsilon\|_{L^\infty} + \sum |a_j| \leq C. \tag{4.1}$$

Proof. It is a classical consequence of the implicit function theorem.
step 1 Decomposition near $\lambda = 1$, $\overline{x} = 0$. We introduce the smooth maps

$$F(v,\mu,x,b_1,\dots,b_n) = \mu^{\frac{2}{p-1}}(\Phi_n + v)(\mu y + x) - \Phi_n - \sum_{j=2}^{n+1} b_j\psi_j$$

and

$$G = ((F,\Lambda\Phi_n),(F,\partial_1\Phi_n),(F,\partial_2\Phi_n),(F,\partial_3\Phi_n),(F,\psi_2),\dots,(F,\psi_{n+1})).$$

We immediately check that $G(\Phi_n,1,0,\dots,0) = 0$ and that

$$\frac{\partial G}{\partial(\mu,x,b_2,\dots,b_{n+1})}_{|(\Phi_n,1,0,\dots,0)}$$

is invertible. In view of the implicit function theorem, for $\kappa > 0$ small enough, for any

$$\|v\|_{L^\infty} \leq \kappa$$

there exists $(\mu, z, a_2, \dots, a_{n+1})$ and

$$\varepsilon = F(v, \mu, z, a_2, \dots, a_{n+1})$$

such that

$$u = \Phi_n + v = \frac{1}{\mu^{\frac{2}{p-1}}}\left(\Phi_n + \sum_{j=2}^{n+1} a_j\psi_j + \varepsilon\right)\left(\frac{x-z}{\mu}\right),$$

$$(\varepsilon, \psi_j) = (\varepsilon, \partial_k\Phi_n) = 0, \quad 1 \le j \le n, \quad 1 \le k \le 3,$$

and there exist two universal constants $K, \tilde{K} > 0$ such that

$$\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \le K\|v\|_{L^\infty}$$

and such that the decomposition is unique under the bound

$$\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \le \tilde{K}. \tag{4.2}$$

step 2 Decomposition near any $\lambda, \overline{x}$. For any $\delta > 0$, we take $C = C(\delta) := K\delta$. Let $u \in X_\delta$ then for some $\lambda' > 0$ and y one has

$$u(x) = \frac{1}{\lambda'^{\frac{2}{p-1}}}(\Phi_n + v)\left(\frac{x-y}{\lambda'}\right), \ \|v\|_{L^\infty} < \delta.$$

The first step then provides the decomposition claimed in the lemma for δ small enough via the formulas $\lambda = \lambda'\mu(v)$, $\overline{x} = y - \lambda' z(v)$, $a_j = a_j(v)$ and $\varepsilon = \varepsilon(v)$. We will show in the next step that the decomposition is unique, implying that the parameters are Fréchet differentiable on X_δ for those of step 1 are.
step 3 Uniqueness of the decomposition. First, from a continuity argument, for any $\epsilon > 0$, there exists $\delta > 0$ such that if

$$(\Phi_n + v)(x) = \frac{1}{\mu^{\frac{2}{p-1}}}(\Phi_n + v')\left(\frac{x-y}{\mu}\right), \ \|v\|_{L^\infty} + \|v'\|_{L^\infty} \le \delta$$

then

$$|\mu - 1| + |y| \le \epsilon.$$

Now recall that $C = K\delta$ and assume that we are given a second decomposition for $u \in X_\delta$. In view of step 2, performing a change of variable, this amount to say that $\Phi_n + v$ admits another decomposition:

$$(\Phi_n + v)(x) = \frac{1}{\overline{\mu}^{\frac{2}{p-1}}}(\Phi_n + \sum_{j=2}^{n+1} \overline{a}_j\psi_j + \overline{\varepsilon})\left(\frac{x-\overline{z}}{\overline{\mu}}\right)$$

and the bound (4.1) gives

$$\sum_{j=2}^{n+1} |\overline{a}_j| + \|\overline{\varepsilon}\|_{L^\infty} \le K\delta.$$

Using the above continuity estimate, one obtains that for δ small enough

$$|\overline{z}| + |\overline{\mu} - 1| \ll \tilde{K}.$$

Therefore, for δ small enough the second decomposition associated with $\overline{\mu}$, $\overline{z}$, $\overline{a}_j$ and $\overline{\varepsilon}$ satisfies (4.2), and is therefore the one given by step 2 by uniqueness. □

Description of the initial datum. We will now focus on solutions of (1.1) that are a suitable perturbation of Φ_n at initial time:

$$u_0 = \frac{1}{\lambda_0^{\frac{2}{p-1}}}(\Phi_n + v_0)\left(\frac{x}{\lambda_0}\right) \tag{4.3}$$

with

$$v_0 = \sum_{j=2}^{n+1} a_j\psi_j + \varepsilon_0, \quad (\varepsilon_0, \psi_j)_\rho = (\varepsilon_0, \partial_k\Phi_n)_\rho = 0, \quad 1 \le j \le n+1, \quad 1 \le k \le 3. \tag{4.4}$$

For $s_0 \gg 1$ and $\mu, K_0 > 0$ three constants to be defined later on, the parameters λ_0, a_j and the profile ε_0 satisfy the bounds

- rescaled solution:

$$\lambda_0 = e^{-s_0}; \tag{4.5}$$

- initial control of the unstable modes:

$$\sum_{j=2}^{n+1} |a_j|^2 \le e^{-2\mu s_0}; \tag{4.6}$$

- smallness of suitable initial norms:

$$\|\varepsilon_0\|_{H^2_\rho} + \|\Delta v_0\|_{L^2} + \|w_0\|_{\dot H^{s_c}} \le K_0 e^{-\mu s_0}; \tag{4.7}$$

where w_0 is given by

$$w_0 = \left(1 - \chi_{\frac{1}{\lambda_0}}\right)\Phi_n + v_0.$$

Note that in view of the L^∞ bound (4.23), the decomposition (4.3) is precisely the one given by Lemma 4.1.

Renormalized flow. As long as the solution $u(t)$ starting from (4.3) belongs to X_δ, Lemma 4.1 applies and it can be written

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(\Phi_n + \psi + \varepsilon)(s,z), \quad y = \frac{x - x(t)}{\lambda(t)} \tag{4.8}$$

with

$$\psi = \sum_{j=2}^{n+1} a_j\psi_j, \quad (\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k\Phi_n)_\rho = 0, \quad 1 \le j \le n+1, \quad 1 \le k \le 3. \tag{4.9}$$

Moreover, as the parameters are Fréchet differentiable in L^∞, and as $u \in C^1((0,T), L^\infty)$ from parabolic regularizing effects, the above decomposition is differentiable with respect to time. We also introduce a further decomposition

$$v = \psi + \varepsilon, \quad \Phi_n + v = \chi_{\frac{1}{\lambda}}\Phi_n + w. \tag{4.10}$$

Consider the renormalized time

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} + s_0.$$

Injecting (4.8) into (1.1) yields the renormalized equation

$$\partial_s\varepsilon + \mathcal{L}_n\varepsilon = F - \mathrm{Mod} \tag{4.11}$$

with the modulation term

$$\mathrm{Mod} = \sum_{j=2}^{n+1}\left[(a_j)_s - \mu_j a_j\right]\psi_j - \left(\frac{\lambda_s}{\lambda} + 1\right)(\Lambda\Phi_n + \Lambda\psi) - \frac{x_s}{\lambda}\cdot(\nabla\Phi_n + \nabla\psi) \tag{4.12}$$

and the force terms

$$F = L(\varepsilon) + \mathrm{NL}, \quad L(\varepsilon) = \left(\frac{\lambda_s}{\lambda} + 1\right)\Lambda\varepsilon + \frac{x_s}{\lambda}\cdot\nabla\varepsilon \tag{4.13}$$

$$\mathrm{NL} = g(\varepsilon+\psi), \quad g(v) = (\Phi_n + v)^p - \Phi_n^p - p\Phi_n^{p-1}v. \tag{4.14}$$

We claim the following bootstrap proposition.

Proposition 4.2 (Bootstrap)**.** *There exist universal constants $0<\mu,\eta\ll 1$, $K\gg 1$ such that for all $s_0 \geq s_0(K,\mu,\eta)\gg 1$ large enough the following holds. For any λ_0 and ε_0 satisfying* (4.5), (4.4) *and*

$$\|(1-\chi_{\frac{1}{\lambda_0}})\Phi_n + \varepsilon_0\|_{\dot H^{s_c}} + \|\varepsilon_0\|_{H^2_\rho} + \|\Delta\varepsilon_0\|_{L^2} \leq e^{-2\mu s_0}, \tag{4.15}$$

there exist $(a_2(0),\dots,a_{n+1}(0))$ satisfying (4.6) *such that the solution starting from u_0 given by* (4.3)*, decomposed according to* (4.8) *satisfies for all $s\geq s_0$:*

- *control of the scaling:*

$$0<\lambda(s)<e^{-\mu s}; \tag{4.16}$$

- *control of the unstable modes:*

$$\sum_{j=2}^{n+1}|a_j|^2 \leq e^{-2\mu s}; \tag{4.17}$$

- *control of the exponentially weighted norm:*

$$\|\varepsilon\|_{H^2_\rho} < Ke^{-\mu s}; \tag{4.18}$$

- *control of a Sobolev norm above scaling:*

$$\|\Delta v\|_{L^2} < Ke^{-\mu s}; \tag{4.19}$$

- *control of the critical norm:*

$$\|w\|_{\dot H^{s_c}} < \eta. \tag{4.20}$$

Proposition 4.2 is the heart of the analysis, and the corresponding solutions are easily shown to satisfy the conclusions of Theorem 1.2. The strategy of the proof follows [10, 42]: we prove Proposition 4.2 by contradiction using a topological argument à la Brouwer: given $(\varepsilon_0,\lambda_0)$ satisfying (4.5), (4.15) and (4.4), we assume that for all $(a_2(0),\dots,a_{n+1}(0))$ satisfying (4.6), the exit time

$$s^* = \sup\{s\geq s_0 \ \text{ such that } \ (4.16),(4.17),(4.18),(4.19),(4.20) \text{ holds on } [s_0,s)\} \tag{4.21}$$

is finite

$$s^* < +\infty \tag{4.22}$$

and look for a contradiction for $0<\mu,\eta,\frac{1}{K}$ small enough and $s_0\geq s_0(K,\mu)$ large enough. From now on, we therefore study the flow on $[s_0,s^*]$ where (4.16), (4.17), (4.18), (4.19) and (4.20) hold. Using a bootstrap method we show that the bounds (4.16), (4.18), (4.19) and (4.20) can be improved, implying that at time s^* necessarily the unstable modes have grown and (4.17) is violated. Since 0 is a linear repulsive equilibrium for these modes, this would contradict Brouwer fixed point theorem.

From the asymptotic (3.5) of ψ_j for $2\leq j\leq n+1$, (4.6) and (4.15), one can fix the constant K_0 independently of $(s_0,\mu,)$ such that (4.7) holds. Also, note that the bootstrap bounds (4.17), (4.18), (4.19) and (4.20) imply the L^∞ bound (4.23), and therefore the decomposition used in the Proposition is well defined since Lemma 4.1 applies.

4.2. L^∞ **bound.** We start with the derivations of *unweighted* L^∞ and Sobolev bounds on v, w which will be essential to control nonlinear terms in the sequel and follow from (4.19), (4.20).

Lemma 4.3 (L^∞ smallness)**.** *There holds*

$$\|v\|_{L^\infty} + \|w\|_{L^\infty} \leq e^{-c\mu s} \leq \eta \ll 1 \tag{4.23}$$

for some universal constants $c > 0$, $0 < \eta \ll 1$.

Proof. We compute from (4.10):

$$w = (1 - \chi_{\frac{1}{\lambda}})\Phi_n + v. \tag{4.24}$$

The self similar decay (1.9) and (4.19) yield:

$$\|w\|_{\dot{H}^2} \lesssim \|v\|_{\dot{H}^2} + \|(1 - \chi_{\frac{1}{\lambda}})\Phi_n\|_{\dot{H}^2} \lesssim K\left[e^{-\mu s} + \lambda(s)^{2-s_c}\right] \leq e^{-c\mu s}.$$

Hence by interpolation using $s_c = \frac{3}{2} - \frac{2}{p-1} < \frac{3}{2} < 2$:

$$\|w\|_{L^\infty} \lesssim \|\hat{w}\|_{L^1} \lesssim \|w\|^{1-\alpha}_{\dot{H}^{s_c}} \|w\|^{\alpha}_{\dot{H}^2}, \quad \alpha = \frac{\frac{3}{2} - s_c}{2 - s_c}$$

which together with (4.20) ensures:

$$\|w\|_{L^\infty} \lesssim e^{-c\mu s}.$$

The decay (1.9) and (4.16), (4.24) yield the L^∞ smallness for v and conclude the proof. $\square$

4.3. **Modulation equations.** We now compute the modulation equations which describe the time evolution of the parameters. They are computed in the self-similar zone, and involve the ρ weighted norm.

Lemma 4.4 (Modulation equations)**.** *There holds the bounds*

$$\left|\frac{\lambda_s}{\lambda} + 1\right| + \left|\frac{x_s}{\lambda}\right| + \sum_{j=2}^{n+1} |(a_j)_s - \mu_j a_j| \lesssim \|\varepsilon\|^2_{H^1_\rho} + \|\Delta v\|^2_{L^2} + \sum_{j=2}^{n+1} |a_j|^2. \tag{4.25}$$

Proof. This lemma is a classical consequence of the choice of orthogonality conditions (4.9), but the control of the nonlinear term relies in an essential way on the L^∞ smallness (4.23).

step 1 Law for a_j. Take the L^2_ρ scalar product of (4.11) with ψ_j for $2 \leq j \leq n+1$, then using (4.9) and the orthogonality

$$(\psi_j, \psi_k)_\rho = \delta_{jk}, \quad \psi_1 = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}, \tag{4.26}$$

we obtain

$$(a_j)_s - \mu_j a_j = \left(\frac{\lambda_s}{\lambda} + 1\right)(\Lambda\psi, \psi_j)_\rho + (F, \psi_j)_\rho.$$

First, from (4.17) one has

$$|(\Lambda\psi, \psi_j)_\rho| \lesssim e^{-\mu s} \ll \eta.$$

We now estimate the F-term given by (4.13) . We use the bound from $p > 5$:

$$\left||1 + z|^p - 1 - pz^{p-1}\right| \lesssim |z|^p + |z|^2$$

to estimate from the L^∞ bound (4.23):

$$|\mathrm{NL}| \lesssim |\varepsilon+\psi|^p + \Phi_n^{p-2}(\varepsilon+\psi)^2 \lesssim (\varepsilon+\psi)^2 = v^2. \tag{4.27}$$

We estimate from the Hardy inequality (A.5):

$$\int \frac{|\nabla v|^2}{1+|y|^2} + \frac{|v|^2}{1+|y|^4} \lesssim \int |\Delta v|^2 + \|v\|^2_{H^1_\rho} \lesssim \int |\Delta v|^2 + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^2 \tag{4.28}$$

and hence using the polynomial bound (3.5):

$$\begin{aligned} |(\mathrm{NL},\psi_j)_\rho| &\lesssim \int v^2|\psi_j|\rho \lesssim \int \frac{|v|^2}{1+|y|^4} \lesssim \int |\Delta v|^2 + \|v\|^2_{H^1_\rho} \\ &\lesssim \|\varepsilon\|^2_{H^1_\rho} + \|\Delta v\|^2_{L^2} + \sum_{j=2}^{n+1}|a_j|^2. \end{aligned}$$

Next, we integrate by parts and use Cauchy Schwarz and (3.5) to estimate:

$$\left|\left(\left(\frac{\lambda_s}{\lambda}+1\right)\Lambda\varepsilon + \frac{x_s}{\lambda}\cdot\nabla\varepsilon, \psi_j\right)_\rho\right| \lesssim \left[\left|\frac{\lambda_s}{\lambda}+1\right| + \left|\frac{x_s}{\lambda}\right|\right] \|\varepsilon\|_{L^2_\rho}$$

and hence the first bound

$$|(a_j)_s - \mu_j a_j| \lesssim \left(\left|\frac{\lambda_s}{\lambda}+1\right| + \left|\frac{x_s}{\lambda}\right|\right)\eta + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}.$$

step 2 Law for scaling and translation. We scalarize (4.11) with $\psi_1 = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}$ and $\frac{\partial_k\Phi_n}{\|\partial_k\Phi_n\|_{L^2}}$ and obtain in a completely similar way

$$\left|\frac{\lambda_s}{\lambda}+1\right| + \left|\frac{x_s}{\lambda}\right| \lesssim \left(\left|\frac{\lambda_s}{\lambda}+1\right| + \left|\frac{x_s}{\lambda}\right|\right)\eta + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}.$$

Summing the above estimates and using the smallness of η yields (4.25). □

4.4. **Energy estimates with exponential weights.** We now turn to the proof of exponential decay which is an elementary consequence of the spectral gap estimate (3.6), the dissipative structure of the flow *and* the L^∞ bound (4.23) to control the non linear term.

Lemma 4.5 (Lyapounov control of exponentially weighed norms)**.** *There holds the differential bound*

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} + c_n\|\varepsilon\|^2_{H^1_\rho} \lesssim \sum_{j=2}^{n+1}|a_j|^4 + \|\Delta v\|^4_{L^2} + \|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2} + \sum_{j=2}^{n+1}|a_j|^2\right], \tag{4.29}$$

$$\begin{aligned} \frac{d}{ds}\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho} + c_n\|\mathcal{L}_n\varepsilon\|^2_{H^1_\rho} &\lesssim \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^4 + \|\Delta v\|^4_{L^2} \\ &+ \|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2} + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2\right], \end{aligned} \tag{4.30}$$

with $c_n>0$ given by (3.6).

Proof. **step 1** L^2 weighted bound. We compute from (4.11):

$$\frac{1}{2}\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} = (\varepsilon, \partial_s\varepsilon)_\rho = -(\mathcal{L}_n\varepsilon, \varepsilon)_\rho + (F - \mathrm{Mod}, \varepsilon)_\rho. \tag{4.31}$$

From (4.12), (4.25):

$$\begin{aligned}|(\varepsilon, \mathrm{Mod})_\rho| \lesssim \|\varepsilon\|_{L^2_\rho}\|\mathrm{Mod}\|_{L^2_\rho} &\lesssim \|\varepsilon\|_{L^2_\rho}\left(\|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}\right)\\ &\lesssim \delta\|\varepsilon\|^2_{L^2_\rho} + C_\delta\left(\|\varepsilon\|^4_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^4 + \|\Delta v\|^4_{L^2}\right)\end{aligned}$$

for any $\delta > 0$. Integrating by parts and using (A.1), we estimate

$$|(\varepsilon, \Lambda\varepsilon)_\rho| + |(\nabla\varepsilon, \varepsilon)_\rho| \lesssim \int(1+|y|^2)\varepsilon^2\rho dy \lesssim \|\varepsilon\|^2_{H^1_\rho} \tag{4.32}$$

from which using (4.25):

$$\left|(L(\varepsilon), \varepsilon)_\rho\right| \lesssim \|\varepsilon\|^2_{H^1_\rho}\left(\|\varepsilon\|^2_{L^2_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}\right).$$

Finally using (4.27), (4.28):

$$\begin{aligned}|(\mathrm{NL}, \varepsilon)_\rho| &\lesssim \int|\varepsilon|v^2\rho dy \leq \delta\int|\varepsilon|^2\rho + C_\delta\int|v|^4\rho dy\\ &\leq \delta\int|\varepsilon|^2\rho + C_\delta\|v\|^2_{L^\infty}\int\frac{|v|^2}{1+|y|^4}dy\\ &\leq \delta\|\varepsilon\|^2_{L^2_\rho} + C_\delta\|v\|^2_{L^\infty}\left[\int|\Delta v|^2 + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2\right].\end{aligned}$$

Injecting the collection of above bounds into (4.31) and using the spectral gap estimate (3.6) with the choice of orthogonality conditions (4.9) yields

$$\begin{aligned}\frac{d}{ds}\|\varepsilon\|^2 \leq\ & -2c_n\|\varepsilon\|_{H^1_\rho}\left(1 - C(\|\varepsilon\|^2_{H^1_\rho} - \sum_{j=2}^{n+1}|a_j|^2 - \|\dagger\Delta v\|^2_{L^2}) - C\delta - C_\delta\|\varepsilon\dagger\|_{H^1_\rho}\right)\\ &+C_\delta\|v\|_{L^\infty}\left[\int\Delta v^2 + \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2\right]\end{aligned}$$

which using the bootstrap bounds (4.17), (4.18) and (4.19) gives (4.29) for s_0 large enough and δ small enough.

step 2 H^2 weighted bound. Let

$$\varepsilon_2 = \mathcal{L}_n\varepsilon,$$

then ε_2 satisfies the orthogonality conditions (4.9):

$$(\varepsilon_2, \psi_j) = (\varepsilon_2, \partial_k\Phi_n) = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3, \tag{4.33}$$

and the equation from (4.11):

$$\partial_s\varepsilon_2 + \mathcal{L}_n\varepsilon_2 = \mathcal{L}_n(F - \mathrm{Mod}).$$

Hence:

$$\frac{1}{2}\frac{d}{ds}\|\varepsilon_2\|^2_{L^2_\rho} = -(\mathcal{L}_n\varepsilon_2,\varepsilon_2)_\rho + (\mathcal{L}_n(F-\mathrm{Mod}),\varepsilon_2)_\rho. \tag{4.34}$$

We estimate from (4.25):

$$\|\mathcal{L}_n\mathrm{Mod}\|_{L^2_\rho} \lesssim \left|\frac{\lambda_s}{\lambda}-1\right| + \left|\frac{x_s}{\lambda}\right| + \sum_{j=2}^{n+1}|(a_j)_s - a_j| \lesssim \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}.$$

We now use the commutator relation

$$[\Delta,\Lambda] = 2\Delta$$

to compute

$$[\mathcal{L}_n,\Lambda] = [-\Delta+\Lambda-p\Phi_n^{p-1},\Lambda] = -2\Delta+p(p-1)\Phi_n^{p-2}r\partial_r\Phi_n = 2(\mathcal{L}_n-\Lambda+p\Phi_n^{p-1})+p(p-1)\Phi_n^{p-2}r\partial_r\Phi_n$$

from which using (4.32), (A.1):

$$\begin{aligned}
|(\varepsilon_2,\mathcal{L}_n\Lambda\varepsilon)_\rho| &= |(\varepsilon_2,[\mathcal{L}_n,\Lambda]\varepsilon)_\rho + (\varepsilon_2,\Lambda\varepsilon_2)_\rho| \\
&\lesssim \|\varepsilon_2\|^2_{H^1_\rho} + |(\varepsilon_2,\Lambda\varepsilon)_\rho| + |(\varepsilon_2,\Phi_n^{p-1}\varepsilon)_\rho| + |(\varepsilon_2,\Phi_n^{p-2}\Lambda\Phi_n\varepsilon)_\rho| \\
&\lesssim \|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho}
\end{aligned}$$

and similarly

$$|(\varepsilon_2,\mathcal{L}_n\partial_k\varepsilon)_\rho| \lesssim \|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho}.$$

Hence from (4.25):

$$|(\varepsilon_2,\mathcal{L}_nL(\varepsilon))_\rho| \lesssim (\|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho})\left(\|\varepsilon\|^2_\rho + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}\right).$$

It remains to estimate the nonlinear term. We first integrate by parts since $\mathcal{L}_n$ is self adjoint for $(\cdot,\cdot)_\rho$ to estimate using the notation (4.14):

$$\begin{aligned}
|(\mathcal{L}_n\mathrm{NL},\varepsilon_2)_\rho| &= \left|(\nabla\mathrm{NL},\nabla\varepsilon_2)_\rho + \left(\frac{2}{p-1}\mathrm{NL} - p\Phi_n^{p-1}\mathrm{NL},\varepsilon_2\right)_\rho\right| \\
&\lesssim |(\nabla g(v),\nabla\varepsilon_2)_\rho| + \left|\left(\frac{2}{p-1}g(v) - p\Phi_n^{p-1}g(v),\varepsilon_2\right)_\rho\right|.
\end{aligned}$$

We now compute explicitly

$$\begin{aligned}
\nabla g(v) &= p\nabla v\left[(\Phi_n+v)^{p-1} - \Phi_n^{p-1}\right] \\
&+ p\nabla\Phi_n\left[(\Phi_n+v)^{p-1} - \Phi_n^{p-1} - (p-1)\Phi_n^{p-2}v\right].
\end{aligned} \tag{4.35}$$

We estimate by homogeneity with the L^∞ bound (4.23):

$$|g(v)| \lesssim |v|^2, \quad |\nabla g(v)| \lesssim |\nabla v||v| + |v|^2$$

and hence the bound using (4.23) again:

$$
\begin{aligned}
& |(\nabla g(v), \nabla \varepsilon_2)_\rho| + \left|\left(\frac{2}{p-1} g(v) - p\Phi_n^{p-1} g(v), \varepsilon_2\right)_\rho\right| \\
\lesssim \; & \int \left[|v||\nabla(v)| + |v|^2\right] |\nabla\varepsilon_2| \rho dy + \int |\varepsilon_2||v|^2 \rho dy \\
\leq \; & \delta\|\varepsilon_2\|^2_{H^1_\rho} + C_\delta \left[\int |v|^2|\nabla v|^2 \rho dy + \int |v|^4 \rho dy\right] \\
\leq \; & \delta\|\nabla\varepsilon_2\|^2_{L^2_\rho} + C_\delta\|v\|^2_{L^\infty}\left[\int \frac{|\nabla v|^2}{1+|y|^2} dy + \int \frac{|v|^2}{1+|y|^4} dy\right] \\
\leq \; & \delta\|\nabla\varepsilon_2\|^2_{L^2_\rho} + C_\delta\|v\|^2_{L^\infty}\left[\|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}\right].
\end{aligned}
$$

The collection of above bounds together with the spectral gap estimate (3.6) and the orthogonality conditions (4.33) injected into (4.34) yields (4.30). □

Remark 4.6. The proof of (4.29) is elementary but requires in an essential way the L^∞ smallness bound[17] (4.23), and in particular the sole control of the H^1_ρ norm cannot suffice to control the nonlinear term $\int |\varepsilon|^{p+1}\rho$ due to both the energy super critical nature of the problem and the exponential weight.

4.5. **Outer global $\dot H^2$ bound.** We recall

$$v = \varepsilon + \psi$$

and now aim at propagating an *unweighted global* $\dot H^2$ decay estimate for v. We have

$$\partial_s v - \Delta v - \frac{\lambda_s}{\lambda}\Lambda v - \frac{x_s}{\lambda}\cdot\nabla v = G$$

with

$$G = \left[\left(\frac{\lambda_s}{\lambda}+1\right)\Lambda\Phi_n + \frac{x_s}{\lambda}\cdot\nabla\Phi_n\right] + \widehat{NL}, \quad \widehat{NL} = (\Phi_n + v)^p - \Phi_n^p.$$

Lemma 4.7 (Global $\dot H^2$ bound). *There holds the Lyapounov type monotonicity formula*

$$\frac{d}{ds}\left[\frac{1}{\lambda^{4-\delta-2s_c}}\int |\Delta v|^2 dy\right] + \frac{1}{\lambda^{4-\delta-2s_c}}\int |\nabla\Delta v|^2 dy \lesssim \frac{1}{\lambda^{4-2s_c-\delta}}\left[\|\varepsilon\|^2_{H^2_\rho} + \sum_{j=2}^{n+1}|a_j|^2\right] \tag{4.36}$$

for some universal constant $0 < \delta \ll 1$.

Proof. We compute the $\dot H^2$ energy identity:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int |\Delta v|^2 dy &= \int \Delta v \Delta\left[\Delta v + \frac{\lambda_s}{\lambda}\Lambda v + \frac{x_s}{\lambda}\cdot\nabla v + G\right] dy \\
&= -\int |\nabla\Delta v|^2 dy + \int \Delta v\Delta\left[\frac{\lambda_s}{\lambda}\Lambda v + \frac{x_s}{\lambda}\cdot\nabla v + G\right] dy
\end{aligned}
$$

and estimate all terms.

[17] or anything above or equal scaling in terms of regularity.

step 1 Parameters terms. For any $\mu>0$, let $v_\mu=\frac{1}{\mu^{\frac{2}{p-1}}}v\left(\frac{y}{\mu}\right)$, then:

$$\int|\Delta v_\mu|^2dy=\frac{1}{\mu^{4-2s_c}}\int|\Delta v|^2dy$$

and hence differentiating and evaluating at $\mu=1$:

$$-2\int\Delta v\Delta(\Lambda v)dy=-(4-2s_c)\int|\Delta v|^2dy.$$

Hence

$$\frac{\lambda_s}{\lambda}\int\Delta v\Delta(\Lambda v)=(2-s_c)\frac{\lambda_s}{\lambda}\int|\Delta v|^2dy.$$

Also, integrating by parts:

$$\int\Delta v\Delta\left(\frac{x_s}{\lambda}\cdot\nabla v\right)dy=0.$$

step 2 G terms. Thanks to the decay of the self similar solution from (1.9):

$$\int|\Delta\Lambda\Phi_n|^2dy+\int|\Delta\nabla\Phi_n|^2dy<+\infty,$$

we estimate in brute force using (4.25) the terms induced by the self similar solution:

$$\begin{aligned}&\left|\int\Delta v\Delta\left\{\left[\left(\frac{\lambda_s}{\lambda}+1\right)\Lambda\Phi_n+\frac{x_s}{\lambda}\cdot\nabla\Phi_n\right]\right\}\right|\\ \lesssim&\left[\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|\right]\|\Delta v\|_{L^2}\le\delta\|\Delta v\|_{L^2}^2+C_\delta\left(\|\varepsilon\|_{H^1_\rho}^2+\|\Delta v\|_{L^2}^2+\sum_{j=2}^{n+1}|a_j|^2\right)^2\\ \le&\ \delta\|\Delta v\|_{L^2}^2+C_\delta\left(\|\varepsilon\|_{H^1_\rho}^2+\sum_{j=2}^{n+1}|a_j|^2\right).\end{aligned}$$

It remains to estimate the nonlinear term. We estimate by homogeneity:

$$\begin{aligned}|\Delta\widehat{NL}|=&\left|p\Delta\Phi_n\left[(\Phi_n+v)^{p-1}-\Phi_n^{p-1}\right]+p(\Phi_n+v)^{p-1}\Delta v\right.\\ +&\ p(p-1)|\nabla\Phi_n|^2\left[(\Phi_n+v)^{p-2}-\Phi_n^{p-2}\right]+p(p-1)|\nabla v|^2(\Phi_n+v)^{p-1}\\ +&\left.2p(p-1)(\Phi_n+v)^{p-2}\nabla\Phi_n\cdot\nabla v\right|\\ \lesssim&\ |\Delta\Phi_n|(|v|^{p-1}+|\Phi_n|^{p-2}|v|)+|\Delta v|(|v|^{p-1}+|\Phi_n|^{p-1})\\ +&\ |\nabla\Phi_n|^2(|v|^{p-2}+|\Phi_n|^{p-3}|v|)+|\nabla v|^2(|v|^{p-1}+|\Phi_n|^{p-1})+|\nabla v||\nabla\Phi_n|(|\Phi_n|^{p-2}+|v|^{p-2})\end{aligned}$$

and hence using the self similar decay of Φ_n and the L^∞ smallness (4.23):

$$\begin{aligned}|\Delta\widehat{NL}|\lesssim&\left[\frac{|\Delta v|}{1+|y|^2}+\frac{|\nabla v|}{1+|y|^3}+\frac{|v|}{1+|y|^4}\right]+\eta\left[|\Delta v|+\frac{|\nabla v|}{1+|y|}+\frac{|v|}{1+|y|^2}\right]\\ +&\ |\nabla v|^2(|v|^{p-1}+|\Phi_n|^{p-1}).\end{aligned}$$

The linear term is estimated using (A.5):

$$\begin{aligned}\int\left|\frac{|\Delta v|}{1+|y|^2}+\frac{|\nabla v|}{1+|y|^3}+\frac{|v|}{1+|y|^4}\right|^2 &\lesssim \frac{1}{A^4}\int_{|y|\geq A}|\Delta v|^2+C_A\|v\|^2_{H^2_\rho}\\ &\leq \delta\int|\Delta v|^2+C_\delta\left(\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right)\end{aligned}$$

and using (A.5) again:

$$\int\left|\eta\left[|\Delta v|+\frac{|\nabla v|}{1+|y|}+\frac{|v|}{1+|y|^2}\right]\right|^2\lesssim \eta\|\Delta v\|^2_{L^2}+\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2.$$

To estimate the nonlinear term, we let

$$q_c=\frac{3(p-1)}{2}\ \text{ so that }\ \dot H^{s_c}\subset L^{q_c}.$$

We estimate using (4.23) with $6(p-2)>q_c$ and Sobolev:

$$\begin{aligned}&\int|\nabla v|^4(|v|^{2(p-2)}+|\Phi_n|^{2(p-2)})\lesssim\|\nabla v\|^4_{L^6}\left[\|v\|^{2(p-2)}_{L^{6(p-2)}}+\|\Phi_n\|^{2(p-2)}_{L^{6(p-2)}}\right]\\ \lesssim\ &\|\Delta v\|^4_{L^2}\left[\|\Phi_n\|^{2(p-2)}_{L^{6(p-2)}}+\|w\|^{2(p-2)}_{L^{6(p-2)}}\right]\lesssim\|\Delta v\|^4_{L^2}\left[1+\|w\|^{\frac{p-1}{2}}_{\dot H^{s_c}}\right]\leq\delta\|\Delta v\|^2_{L^2}.\end{aligned}$$

We have therefore obtained

$$\int|\Delta\widehat{NL}|^2\leq\delta\|\Delta v\|^2_{L^2}+C_\delta\left(\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right).$$

The collection of above bounds and (4.25) yields (4.36). □

4.6. **Control of the critical norm.** We now claim the control of the critical norm of w (defined by (4.10)).

Lemma 4.8 (Control of the critical norm)**.** *There holds the Lyapounov type control*

$$\frac{d}{ds}\int|\nabla^{s_c}w|^2dy+\int|\nabla^{s_c+1}w|^2dy\lesssim\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\lambda^{\delta(2-s_c)}+\|\Delta v\|^\delta_{L^2}.\quad(4.37)$$

for some small enough universal constant $0<\delta=\delta(p)\ll1$.

Proof. Let

$$\widetilde{\Phi_n}=\chi_{\frac{1}{\lambda}}\Phi_n,\quad(4.38)$$

we compute the evolution equation of w:

$$\partial_s w-\Delta w=\frac{\lambda_s}{\lambda}\Lambda w+\frac{x_s}{\lambda}\cdot\nabla w+\widetilde G\quad(4.39)$$

with

$$\begin{aligned}\widetilde G&=\left(\frac{\lambda_s}{\lambda}+1\right)\chi_{\frac1\lambda}\Lambda\Phi_n+\frac{x_s}{\lambda}\cdot\nabla\widetilde\Phi_n+2\nabla\chi_{\frac1\lambda}\cdot\nabla\Phi_n+\Delta\chi_{\frac1\lambda}\Phi_n-(\chi_{\frac1\lambda}-\chi^p_{\frac1\lambda})\Phi^p_n+\widetilde{\mathrm{NL}},\\ \widetilde{\mathrm{NL}}&=(\widetilde{\Phi_n}+w)^p-(\widetilde{\Phi_n})^p.\end{aligned}$$

Observe from the space localization of the cut, from the decay of the self similar solution, and from (4.19) and (4.20):

$$\forall s_c\leq s\leq 2,\quad\|w\|_{\dot H^s}\lesssim\eta.\quad(4.40)$$

We compute:

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int|\nabla^{s_c}w|^2dy &= \int\nabla^{s_c}w\cdot\nabla^{s_c}\left[\Delta w+\frac{\lambda_s}{\lambda}\Lambda w+\frac{x_s}{\lambda}\cdot\nabla w+\widetilde{G}\right]dy\\ &= -\int|\nabla^{s_c+1}w|^2+\int\nabla^{s_c}w\cdot\nabla^{s_c}\left[\frac{\lambda_s}{\lambda}\Lambda w+\frac{x_s}{\lambda}\cdot\nabla w+\widetilde{G}\right]dy\end{aligned}$$

and estimate all terms.

step 1 Parameters terms. For any $\mu>0$, let $w_\mu=\frac{1}{\mu^{\frac{2}{p-1}}}w\left(\frac{y}{\mu}\right)$, then :

$$\int|\nabla^{s_c}w_\mu|^2dy=\int|\nabla^{s_c}w|^2dy$$

and hence differentiating at $\mu=1$:

$$-2\int\nabla^{s_c}w\cdot\nabla^{s_c}(\Lambda w)dy=0.$$

Integrating by parts:

$$\int\nabla^{s_c}w\cdot\nabla^{s_c}\left(\frac{x_s}{\lambda}\cdot\nabla w\right)dy=0.$$

step 2 $\widetilde{G}$ terms. The decay of the self similar solution and the space localization of the cut ensure using $1<s_c<2$:

$$\begin{aligned}&\left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n+\Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^{s_c}}\\ \lesssim\ &\left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n+\Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^1}^{2-s_c}\left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n+\Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^2}^{s_c-1}\\ \lesssim\ &\left(\frac{\lambda^2}{\lambda^{s_c-1}}\right)^{2-s_c}\left(\frac{\lambda^2}{\lambda^{s_c-2}}\right)^{s_c-1}\lesssim\lambda^2,\end{aligned}$$

and similarly

$$\begin{aligned}&\left\|\left(\chi_{\frac{1}{\lambda}}-\chi^p_{\frac{1}{\lambda}}\right)\Phi_n^p\right\|_{\dot{H}^{s_c}}\lesssim\left\|\left(\chi_{\frac{1}{\lambda}}-\chi^p_{\frac{1}{\lambda}}\right)\Phi_n^p\right\|_{\dot{H}^1}^{2-s_c}\left\|\left(\chi_{\frac{1}{\lambda}}-\chi^p_{\frac{1}{\lambda}}\right)\Phi_n^p\right\|_{\dot{H}^2}^{s_c-1}\\ \lesssim\ &(\lambda^{3-s_c})^{2-s_c}(\lambda^{4-s_c})^{s_c-1}\lesssim\lambda^2.\end{aligned}$$

Using (4.25):

$$\begin{aligned}&\left\|\left(\frac{\lambda_s}{\lambda}+1\right)\chi_{\frac{1}{\lambda}}\Lambda\Phi_n+\frac{x_s}{\lambda}\cdot\nabla(\chi_{\frac{1}{\lambda}}\Phi_n)\right\|_{\dot{H}^{s_c}}\\ \lesssim\ &\left|\frac{x_s}{\lambda}\right|+\left|\frac{\lambda_s}{\lambda}+1\right|\lesssim\|\varepsilon\|^2_{L^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\|\Delta v\|^2_{L^2}.\end{aligned}$$

We now turn to the control of the nonlinear term and claim the bound:

$$\|\nabla^{s_c}\widetilde{NL}\|_{L^2}\lesssim\|\nabla^{s_c+\alpha}w\|_{L^2}\tag{4.41}$$

for some small enough universal constant $0<\alpha=\alpha(p)\ll1$. Assume (4.41), we then interpolate with $\delta=\frac{\alpha}{2-s_c}$ and use (4.24), (4.20) and the decay of the self similar solution to estimate:

$$\|\nabla^{s_c+\alpha}w\|_{L^2}\lesssim\|\nabla^{s_c}w\|_{L^2}^{1-\delta}\|\Delta w\|^\delta_{L^2}\lesssim\lambda^{\delta(2-s_c)}+\|\Delta v\|^\delta_{L^2},$$

and the collection of above bounds yields (4.37).

Proof of (4.41). We compute

$$\begin{aligned}
\nabla\widetilde{NL} &= p\nabla(\widetilde{\Phi_n}+w)(\widetilde{\Phi_n}+w)^{p-1}-p\nabla\widetilde{\Phi_n}\widetilde{\Phi_n}^{p-1}\\
&= p\nabla\widetilde{\Phi_n}\left[(\widetilde{\Phi_n}+w)^{p-1}-\widetilde{\Phi_n}^{p-1}\right]+p\nabla w(\widetilde{\Phi_n}+w)^{p-1}\\
&= pg_1(w)\nabla(\widetilde{\Phi_n}+w)+p\widetilde{\Phi_n}^{p-1}\nabla w
\end{aligned}$$

with

$$g_1(w)=(\widetilde{\Phi_n}+w)^{p-1}-\widetilde{\Phi_n}^{p-1}.$$

Hence letting

$$s_c=1+\nu,\quad 0<\nu=\frac{1}{2}-\frac{2}{p-1}<\frac{1}{2},$$

we estimate:

$$\|\nabla^{s_c}\widetilde{\mathrm{NL}}\|_{L^2}\lesssim\left\|\nabla^\nu\left[g_1(w)\nabla(\widetilde{\Phi_n}+w)\right]\right\|_{L^2}+\left\|\nabla^\nu\left(\widetilde{\Phi_n}^{p-1}\nabla w\right)\right\|_{L^2}. \tag{4.42}$$

For the first term, we use the following commutator estimate proved in Appendix B: let

$$0<\nu<1,\quad 1<p_1,p_2,p_3,p_4<+\infty,\quad \frac{1}{2}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$$

then

$$\|\nabla^\nu(uv)\|_{L^2}\lesssim\|u\|_{\dot B^\nu_{p_1,2}}\|v\|_{L^{p_2}}+\|u\|_{L^{p_4}}\|v\|_{\dot B^\nu_{p_3,2}}, \tag{4.43}$$

where we use here the standard space formulation of Besov norms for $0<s<1$ and $1\le p<+\infty$[18]:

$$\|u\|_{\dot B^s_{p,2}}\sim\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|u(\cdot-y)-u(\cdot)\|_{L^p}}{t^s}\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}. \tag{4.44}$$

We pick a small enough $0<\alpha\ll 1$ to be chosen later and

$$\frac{1}{p_1}=\frac{1}{3}+\frac{\alpha}{3},\quad \frac{1}{p_2}=\frac{1}{6}-\frac{\alpha}{3}$$
$$\frac{1}{p_3}=\frac{1+\alpha+\nu}{3},\quad \frac{1}{p_4}=\frac{1-2(\alpha+\nu)}{6}.$$

Observe that

$$-\nu+\frac{3}{p_2}=\frac{3}{p_4}$$

and hence from (4.43), the embedding of $\dot H^{s,p}$ in $\dot B^s_{p,2}$, and Sobolev[19]:

$$\begin{aligned}
&\left\|\nabla^\nu\left[g_1(w)\nabla(\widetilde{\Phi_n}+w)\right]\right\|_{L^2}\\
\lesssim\;&\|\nabla(\widetilde{\Phi_n}+w)\|_{L^{p_1}}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}+\|\nabla(\widetilde{\Phi_n}+w)\|_{\dot B^\nu_{p_3,2}}\|g_1(w)\|_{L^{p_4}}\\
\lesssim\;&\|\nabla^{1+\frac{3}{2}-\frac{3}{p_1}}(\widetilde{\Phi_n}+w)\|_{L^2}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}+\|\nabla^{1+\nu+\frac{3}{2}-\frac{3}{p_3}}(\widetilde{\Phi_n}+w)\|_{L^2}\|\nabla^\nu g_1(w)\|_{L^{p_2}}\\
\lesssim\;&\|\nabla^{\frac{3}{2}-\alpha}(\widetilde{\Phi_n}+w)\|_{L^2}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}.
\end{aligned}$$

Since $s_c=\frac{3}{2}-\frac{2}{p-1}<\frac{3}{2}$, we may pick $0<\alpha\ll 1$ with $\frac{3}{2}-\alpha>s_c$ and hence using (4.40) and the decay of the self similar solution:

$$\|\nabla^{\frac{3}{2}-\alpha}(\widetilde{\Phi_n}+w)\|_{L^2}\lesssim 1.$$

[18] see for example [4].

[19] using $\frac{3}{2}-\frac{3}{p_3}=\frac{1}{2}-(\alpha+\nu)>0$.

Let now

$$f(z) = (1+z)^{p-1} - 1$$

then $f(0) = 0$ and

$$|f(z_2) - f(z_1)| = \left|\int_{z_1}^{z_2} f'(\tau)d\tau\right| \lesssim \int_{z_1}^{z_2} (1+|\tau|^{p-2})d\tau \lesssim |z_2 - z_1|(1+|z_1|^{p-2}+|z_2|^{p-2})$$

and hence by homogeneity:

$$|g_1(w_2) - g_1(w_1)| \lesssim |w_2 - w_1|(|\widetilde{\Phi_n}|^{p-2} + |w_2|^{p-2} + |w_1|^{p-2}).$$

Using the L^∞ bound (4.23), (4.44), and Sobolev[20]

$$\begin{aligned}
\|g_1(w)\|_{\dot{B}^{\nu}_{p_2,2}} &\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|g_1(w(\cdot - y)) - g_1(w(\cdot))\|_{L^{p_2}}}{t^\nu}\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|w(\cdot - y) - w(\cdot)\|_{L^{p_2}}}{t^\nu}\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}} \sim \|w\|_{\dot{B}^{\nu}_{p_2,2}} \\
&\lesssim \|\nabla^{\nu+\frac{3}{2}-\frac{3}{p_2}} w\|_{L^2} = \|\nabla^{s_c+\alpha} w\|_{L^2}.
\end{aligned}$$

The collection of above bounds yields the control of the first term of (4.42):

$$\|\nabla^\nu \left[g_1(w)\nabla(\widetilde{\Phi_n} + w)\right]\|_{L^2} \lesssim \|\nabla^{s_c+\alpha} w\|_{L^2}.$$

For the second term in (4.42), we recall the following estimate proved in [42]: let $0 < \nu < 1$ and $\mu > 0$ with $\mu + \nu < \frac{3}{2}$, let f smooth radially symmetric with

$$|\partial_r^k f| \lesssim \frac{1}{1+r^{\mu+k}}, \quad k = 0, 1, \tag{4.45}$$

then there holds the generalized Hardy bound

$$\|\nabla^\nu (uf)\|_{L^2} \lesssim \|\nabla^{\nu+\mu} f\|_{L^2}. \tag{4.46}$$

We then pick again a small enough $0 < \alpha \ll 1$ and let

$$\mu = \alpha, \quad \mu + \nu = \nu + \alpha = s_c - 1 + \alpha < \frac{3}{2}$$

for $0 < \alpha \ll 1$ small enough, and $f = (\chi_{\frac{1}{\lambda}} \Phi_n)^{p-1}$ satisfies

$$|\partial_r^k f| \lesssim \frac{1}{1+r^{2+k}} \lesssim \frac{1}{1+r^{\mu+k}}.$$

Hence

$$\|\nabla^\nu \left(\widetilde{\Phi_n}^{p-1} \nabla w\right)\|_{L^2} \lesssim \|\nabla^{\nu+\mu+1} w\|_{L^2} = \|\nabla^{s_c+\alpha} w\|_{L^2}.$$

This concludes the proof of (4.41). □

[20] Here we use that $\dot{B}^s_{2,2}$ embeds in $\dot{B}^t_{p,2}$ with $s - 3/2 = t - 3/p$ for $p \geq 2$, and $\dot{B}^s_{2,2} = \dot{H}^s$.

4.7. **Conclusion.** We are now in position to conclude the proof of Proposition 4.2 which then easily implies Theorem 1.2.

Proof of Proposition 4.2. We recall that we are arguing by contradiction assuming (4.22). We first show that the bounds (4.16), (4.18), (4.19) and (4.20) can be improved on $[s_0, s^*]$, and then, the existence of the data $(a_j(0))_{2\leq j\leq n+1}$ follows from a classical topological argument à la Brouwer.

step 1 Improved scaling control. We estimate from (4.17), (4.18), (4.19), (4.25):

$$\left|\frac{\lambda_s}{\lambda}+1\right| \lesssim K^2 e^{-2\mu s} \tag{4.47}$$

and hence after integration:

$$\left|\log\left(\frac{\lambda(s)}{\lambda_0}\right)+s-s_0\right| \lesssim \int_{s_0}^{+\infty} K^2 e^{-2\mu\tau}d\tau \lesssim 1+o(1)$$

for s_0 large enough, which together with (4.5) implies:

$$\lambda(s) = \left(\lambda(s_0)e^{s_0}\right)e^{-s}(1+o(1)) \ \text{ and hence } \ \frac{e^{-s}}{2} \leq \lambda(s) \leq 2e^{-s}. \tag{4.48}$$

step 2 Improved Sobolev bounds.

L^2_ρ bound. From (4.29), (4.17), (4.19), (4.23):

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} + c_n\|\varepsilon\|^2_{H^1_\rho} \lesssim (1+K^4)e^{-4\mu s} + K^2 e^{-2\mu s}e^{-2c\mu s} \leq e^{-(2+c)\mu s}$$

for $s \geq s_0$ large enough. From now on, we may fix once and for all the value

$$\mu = \frac{c_n}{4} \tag{4.49}$$

and hence

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} + 4\mu\|\varepsilon\|^2_{H^1_\rho} \leq e^{-(2+c)\mu s} \tag{4.50}$$

which time integration yields using (4.7):

$$\begin{aligned}\|\varepsilon(s)\|^2_{L^2_\rho} + 2\mu e^{-2\mu s}\int_{s_0}^{s} e^{2\mu\sigma}\|\varepsilon\|^2_{H^1_\rho}d\sigma &\leq \left(e^{2\mu s_0}\|\varepsilon(s_0)\|^2_{L^2_\rho}\right)e^{-2\mu s} + e^{-2\mu s}\int_{s_0}^{s} e^{-\mu c\tau}d\tau \\ &\lesssim K_0^2 e^{-2\mu s}.\end{aligned} \tag{4.51}$$

H^2_ρ bound. We estimate from (4.30) like for the proof of (4.50):

$$\frac{d}{ds}\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho} + 4\mu\|\mathcal{L}_n\varepsilon\|^2_{H^1_\rho} \lesssim \|\varepsilon\|^2_{H^1_\rho} + e^{-(2+c)\mu s}$$

whose time integration with the initial bound (4.7) and the bound (4.51) ensures:

$$\|\mathcal{L}_n\varepsilon(s)\|^2_{L^2_\rho} \lesssim K_0^2 e^{-2\mu s}.$$

We recall

$$(\mathcal{L}_n\varepsilon,\varepsilon)_\rho = \|\nabla\varepsilon\|^2_{L^2_\rho} + \int\left(\frac{2}{p-1} - p\Phi_n^{p-1}\right)|\varepsilon|^2\rho dy$$

and hence we first estimate from the spectral bound (3.6), the orthogonality conditions (4.9), and Cauchy-Schwarz:

$$\|\nabla\varepsilon\|^2_{L^2_\rho} \leq (\mathcal{L}_n\varepsilon,\varepsilon)_\rho + C\|\varepsilon\|^2_{L^2_\rho} \lesssim \|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho} + \|\varepsilon\|^2_{L^2_\rho} \lesssim K_0^2 e^{-2\mu s}. \tag{4.52}$$

This yields using (A.2):

$$\|\varepsilon\|^2_{H^2_\rho} \lesssim \|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho} + \|\varepsilon\|^2_{H^1_\rho} \tag{4.53}$$

and hence the improved bound

$$\|\varepsilon\|_{H^2_\rho}^2 \lesssim K_0^2 e^{-2\mu s}. \tag{4.54}$$

$\dot{H}^2$ *bound.* We rewrite (4.36) using (4.17), (4.25), (4.54)

$$\frac{d}{ds}\|\Delta v\|_{L^2}^2 + (4-\delta-2s_c)\|\Delta v\|_{L^2}^2 \lesssim K_0^2 e^{-2\mu s}.$$

By possibly diminishing the value of c_n, we may always assume

$$4-\delta-2s_c > c_n = 4\mu$$

and hence from (4.7):

$$\|\Delta v\|_{L^2}^2 \leq K_0^2 e^{-4\mu s} e^{4\mu s_0} e^{-2\mu s_0} + e^{-4\mu s}\int_{s_0}^{s} K_0^2 e^{4\mu\tau} e^{-2\mu\tau} d\tau \lesssim K_0^2 e^{-2\mu s}. \tag{4.55}$$

$\dot{H}^{s_c}$ *bound.* We now rewrite (4.37) using (4.16)-(4.20):

$$\frac{d}{ds}\int \|\nabla^{s_c} w\|_{L^2}^2 \leq e^{-c\mu s}$$

for some universal constant $c > 0$ which time integration using (4.7) ensures:

$$\|\nabla^{s_c} w(s)\|_{L^2}^2 \lesssim \|\nabla^{s_c} w(s_0)\|_{L^2}^2 + e^{-cs_0} < \frac{\eta}{2} \tag{4.56}$$

for s_0 large enough.

step 3 The Brouwer fixed point argument. We conclude from (4.48), (4.54), (4.55), (4.56), the definition (4.21) of s^* and a simple continuity argument that the contradiction assumption (4.22) implies from (4.17):

$$\sum_{j=2}^{n+1} |a_j(s^*)|^2 = e^{-2\mu s^*}. \tag{4.57}$$

Moreover, the vector field is strictly outgoing from (4.25), (4.17), (4.18), (4.19):

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\sum_{j=2}^{n+1}|a_j e^{\mu s}|^2 &= \sum_{j=2}^{n+1} a_j e^{2\mu s}((a_j)_s + \mu a_j) = \sum_{j=2}^{n+1} a_j e^{2\mu s}\left[(\mu+\mu_j)a_j + O\left(K^2 e^{-2\mu s}\right)\right] \\ &\geq \mu \sum_{j=2}^{n+1}|a_j e^{\mu s}|^2 + O\left(K^2 e^{-\mu s}\right)\end{aligned}$$

from which

$$\left(\frac{d}{ds}\sum_{j=2}^{n+1}|a_j e^{\mu s}|^2\right)(s^*) > \mu + O(K^2 e^{-\mu s_0}) > 0$$

for s_0 large enough. We conclude from standard argument that the map

$$(a_j(0)e^{\mu s_0})_{2\leq j\leq n+1} \mapsto (a_j(s^*)e^{\mu s^*})_{2\leq j\leq n+1}$$

is continuous in the unit ball of $\mathbb{R}^n$, and the identity on its boundary, a contradiction to Brouwer's theorem. This concludes the proof of Proposition 4.2. □

We are now in position to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let an initial data as in Proposition 4.2, then the corresponding solution $u(s,y)$ admits on $[s_0,+\infty)$ a decomposition (4.8) with the bounds (4.17), (4.23), (4.19), (4.20), (4.48).

step 1 Self similar time blow up. Using (4.48), the life space of the solution u is finite

$$T=\int_{s_0}^{+\infty}\lambda^2(s)ds\lesssim\int_{s_0}^{+\infty}e^{-2s}ds<+\infty,$$

and hence

$$T-t=\int_{s}^{+\infty}\lambda^2(s)ds\sim e^{-2s}.$$

We may therefore rewrite (4.47):

$$|\lambda\lambda_t+1|\lesssim(T-t)^{\mu}$$

and integrating in time using $\lambda(T)=0$ yields

$$\lambda(t)=\sqrt{(2+o(1))(T-t)}. \tag{4.58}$$

Also from (4.25):

$$\int_0^T|x_t|=\int_{s_0}^{+\infty}|x_s|ds\lesssim\int_{s_0}^{+\infty}e^{-s-2\mu s}ds<+\infty$$

and (1.12) is proved.

step 2 Asymptotic stability above scaling. We now prove (1.13) and (1.15). We first estimate from (4.24) using the self similar decay of Φ_n:

$$\begin{aligned}\|w\|_{\dot H^2} &\lesssim \|v\|_{\dot H^2}+\|(1-\chi_{\frac{1}{\lambda}})\Phi_n\|_{\dot H^2}\lesssim e^{-2\mu s}+\lambda^{2-s_c}(s)\\ &\to 0 \ \text{ as } \ t\to T.\end{aligned}$$

Hence from (4.20):

$$\forall s_c<\sigma\le 2,\quad \lim_{s\to+\infty}\|w(s)\|_{\dot H^\sigma}=0$$

which using (4.24) and the self similar decay of Φ_n again implies

$$\forall s_c<\sigma\le 2,\quad \lim_{s\to+\infty}\|v(s)\|_{\dot H^\sigma}=0,$$

this is (1.13). At the critical level, we have from (4.8), (4.10) and the sharp self similar decay from Proposition 2.2:

$$\|u(t)\|_{\dot H^{s_c}}=\|\chi_{\frac{1}{\lambda}}\Phi_n+w\|_{\dot H^{s_c}}=c_n(1+o(1))\sqrt{|\log\lambda|},\quad c_n\neq 0,$$

and (4.58) now yields (1.15).

step 3 Boundedness below scaling. We now prove (1.14).
Control of the Dirichlet energy. Recall the notation (4.38) and compute by rescaling using the self similar decay of Φ_n:

$$\lambda^{2(s_c-1)}\left[\|\nabla\widetilde{\Phi_n}\|_{L^2}^2+\|\widetilde{\Phi_n}\|_{L^{p+1}}^{p+1}\right]\lesssim 1.$$

Hence the dissipation of energy which is translation invariant ensures

$$\begin{aligned}\lambda^{2(s_c-1)}\|\nabla w\|_{L^2}^2 &\lesssim \lambda^{2(s_c-1)}\left[\|\nabla(\widetilde{\Phi_n}+w)\|_{L^2}^2+\|\nabla\widetilde{\Phi_n}\|_{L^2}^2\right]\lesssim 1+2E(u)+\frac{2}{p+1}\|u\|_{L^{p+1}}^{p+1}\\ &\lesssim 1+|E_0|+\lambda^{2(s_c-1)}\|w\|_{L^{p+1}}^{p+1}.\end{aligned}$$

We now interpolate using the smallness[21] (4.20)

$$\|w\|_{L^{p+1}}^{p+1}\lesssim \|w\|_{\dot H^{s_c}}^{p-1}\|\nabla w\|_{L^2}^2\lesssim \eta\|\nabla w\|_{L^2}^2$$

and hence

$$\lambda^{2(s_c-1)}\|\nabla w\|_{L^2}^2\lesssim C(u_0) \tag{4.59}$$

and

$$\|\nabla u\|_{L^2}^2\lesssim \lambda^{2(s_c-1)}\left[\|\nabla\widetilde{\Phi_n}\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right]\lesssim 1.$$

Proof of (1.14). Let now $1\leq\sigma<s_c$, then using (4.20), (4.59) and interpolation:

$$\begin{aligned}\|\nabla^\sigma u\|_{L^2} &\lesssim \lambda^{s_c-\sigma}\|\nabla^\sigma\widetilde{\Phi_n}\|_{L^2}+\lambda^{s_c-\sigma}\|\nabla^\sigma w\|_{L^2}\lesssim 1+\lambda^{s_c-\sigma}\|\nabla w\|_{L^2}^{\frac{s_c-\sigma}{s_c-1}}\|\nabla^{s_c}w\|_{L^2}^{\frac{\sigma-1}{s_c-1}}\\ &\lesssim 1+\left(\lambda^{s_c-1}\|\nabla w\|_{L^2}\right)^{\frac{s_c-\sigma}{s_c-1}}\lesssim C(u_0)\end{aligned}$$

and (1.14) is proved. This concludes the proof of Theorem 1.2. □

4.8. **The Lipschitz dependence.**

We now state the Lipschitz aspect of the set of solutions constructed in this paper.

Proposition 4.9 (Lipschitz dependence)**.** *Let $s_0\gg1$, $\varepsilon_0^{(1)}$ and $\varepsilon_0^{(2)}$ satisfy* (4.4) *and* (4.15)*, and take $\lambda_0^{(1)}=\lambda_0^{(2)}=e^{-s_0}$. Then the parameters $(a_j^{(1)}(0))_{2\leq j\leq n+1}$ and $(a_j^{(2)}(0))_{2\leq j\leq n+1}$, associated by Proposition 4.2 to $(\varepsilon^{(1)},\lambda_0^{(1)})$ and $(\varepsilon^{(2)},\lambda_0^{(2)})$ respectively, satisfy:*

$$\sum_{j=2}^{n+1}\left|a_j^{(1)}(0)-a_j^{(2)}(0)\right|^2\lesssim\left\|\varepsilon_0^{(1)}-\varepsilon_0^{(2)}\right\|_{L^2_\rho}^2. \tag{4.60}$$

Proof. The idea of the proof is classical, see for instance [17]. We study the difference of two solutions, and use the bounds we already derived in the existence result as a priori bounds now. This allows us to control the difference of solutions at a low regularity level which is sufficient to conclude.

We use the superscripts (i), $i=1,2$ for all variables associated to the two solutions respectively: $u^{(i)}$ for (4.8), $v^{(i)}$ for (4.10), $\psi^{(i)}$ for (4.9), $\lambda^{(i)}$ for the scales and $x^{(i)}$ for the central points. The differences are denoted by

$$\triangle\varepsilon:=\varepsilon^{(1)}-\varepsilon^{(2)},\ \triangle a_j:=a_j^{(1)}-a_j^{(2)},\ \triangle v:=v^{(1)}-v^{(2)}.$$

We compare the two renormalized solutions at the same renormalized time s. The time evolution for the difference is given by

$$\begin{aligned}\triangle\varepsilon_s+\mathcal{L}_n\triangle\varepsilon &= \tfrac{d}{ds}\left[\log\left(\tfrac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\Lambda(\Phi_n+v^{(2)})+\left(\tfrac{x_s^{(1)}}{\lambda^{(1)}}-\tfrac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla(\Phi_n+v^{(2)})\\ &\quad-\sum_{j=2}^{n+1}(\triangle a_{j,s}-\mu_j\triangle a_j)\psi_j+\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)\Lambda\triangle v\\ &\quad+\tfrac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v+\left[(\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v\right].\end{aligned} \tag{4.61}$$

[21] this is the only place in the proof where we use that the critical norm is small, bounded suffices everywhere else.

step 1 Modulation equations. We claim that

$$\begin{array}{rl} & \left|\frac{d}{ds}\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right| + \left|\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right| + \sum_{j=2}^{n+1} |\triangle a_{j,s} - \mu_j \triangle a_j| \\ \lesssim & e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1}|\triangle a_j|\right). \end{array} \quad (4.62)$$

We now show this estimate. Taking the scalar product of (4.61) with $\psi_1 = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}$, using the orthogonality conditions (4.9) and (4.26) and the fact that ψ_j is radial for $1 \leq j \leq n+1$, yields the identity

$$\begin{array}{rl} & \frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right](\Lambda(\Phi_n + v^{(2)}), \psi_1)_\rho \\ = & -\left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla\varepsilon^{(2)}, \psi_1\right)_\rho - \left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}} + 1\right)(\Lambda\triangle v, \psi_1)_\rho - \left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle\varepsilon, \psi_1\right)_\rho \\ & -\left((\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1}\triangle v, \psi_1\right)_\rho \end{array} \quad (4.63)$$

and we now estimate each term. The coercivity (A.1) and the bounds (4.17) and (4.18) yields

$$(\Lambda(\Phi_n + v^{(2)}), \psi_1)_\rho = 1 + O(e^{-\mu s}),$$

$$\left|\left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla\varepsilon^{(2)}, \psi_1\right)_\rho\right| \lesssim e^{-\mu s}\left|\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right|.$$

The modulation estimate (4.25), with (4.17), (4.18) and (4.19) and an integration by parts yields

$$\left|\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}} + 1\right)(\Lambda\triangle v, \psi_1)_\rho - \left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle\varepsilon, \psi_1\right)_\rho\right| \lesssim e^{-\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1}|\triangle a_j|\right).$$

Eventually, for the difference of the nonlinear terms the nonlinear inequality

$$\left|(x+y)^p - (x+z)^p - px^{p-1}(y-z)\right| \lesssim (|x|^{p-2} + |y|^{p-2} + |z|^{p-2})(|y| + |z|)|y-z|$$

for any x, y, z and the bound (4.23) yields the pointwise estimate

$$\left|(\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1}\triangle v\right| \lesssim e^{-c\mu s}|\triangle v|, \quad (4.64)$$

which implies

$$\left|\left((\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1}\triangle v, \psi_1\right)_\rho\right| \lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1}|\triangle a_j|\right). \quad (4.65)$$

The collection of the above bounds, when plugged in (4.63), yields

$$\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right| \lesssim e^{-\mu s}\left|\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right| + e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1}|\triangle a_j|\right).$$

With the same techniques, taking the scalar product of (4.61) with $\partial^k\Phi_n$, $k=1,2,3$ implies

$$\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|\lesssim e^{-\mu s}\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right|+e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

The two above equations, when put together, imply the estimate

$$\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right|+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|\lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

The corresponding estimate for $|\triangle a_{j,s}+\mu_j\triangle a_j|$ follows along the same lines, and therefore (4.62) is proven.

step 2 Localized energy estimate. We claim the differential bound

$$\frac{d}{ds}\|\triangle\varepsilon\|^2_{L^2_\rho}+c_n\|\triangle\varepsilon\|^2_{L^2_\rho}\lesssim e^{-c\mu s}\sum_{j=2}^{n+1}|\triangle a_j|^2 \tag{4.66}$$

which we now prove. From the evolution equation (4.61) and the orthogonality conditions (4.9) one obtains first the identity

$$\begin{aligned}\frac{d}{ds}\frac12\|\triangle\varepsilon\|^2_{L^2_\rho} \;=\;& -(\mathcal{L}_n\triangle\varepsilon,\triangle\varepsilon)_\rho+\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right](\Lambda v^{(2)},\triangle\varepsilon)_\rho\\ &+\left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla v^{(2)},\triangle\varepsilon\right)_\rho+\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)(\Lambda\triangle v,\triangle\varepsilon)_\rho\\ &+\left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v,\triangle\varepsilon\right)_\rho+\left((\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v,\triangle\varepsilon\right)_\rho\end{aligned} \tag{4.67}$$

and we now estimate each term. The spectral gap (3.6) and (4.9) imply

$$-(\mathcal{L}_n\triangle\varepsilon,\triangle\varepsilon)_\rho\le -c_n\|\triangle\varepsilon\|^2_{L^2_\rho}.$$

The modulation estimates (4.62) of step 1 and Cauchy-Schwarz imply

$$\begin{aligned}&\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right](\Lambda v^{(2)},\triangle\varepsilon)_\rho+\left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla v^{(2)},\triangle\varepsilon\right)_\rho\right|\\ \lesssim\;&\left(\left|\frac{d}{ds}\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right|\|\Lambda v^{(2)}\|_{L^2_\rho}+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|\|\nabla v^{(2)}\|_{L^2_\rho}\right)\|\triangle\varepsilon\|_{L^2_\rho}\\ \lesssim\;&\|v^{(2)}\|_{H^2_\rho}e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right)\|\triangle\varepsilon\|_{L^2_\rho}\\ \lesssim\;&e^{-(1+c)\mu s}\left(\|\triangle\varepsilon\|^2_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|^2\right)\end{aligned}$$

where we used (A.1), (4.17) and (4.18) to control $v^{(2)}$. Using the modulation estimate (4.25), with (4.17), (4.18) and (4.19) for $u^{(1)}$, integrating by parts and applying

Cauchy-Schwarz and (A.1) yields

$$
\begin{aligned}
& \left|\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)(\Lambda\triangle v,\triangle\varepsilon)_\rho+\left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v,\triangle\varepsilon\right)_\rho\right| \\
\lesssim\ & \left(\left|\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right|+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}\right|\right)\left(|(\Lambda\triangle\psi,\triangle\varepsilon)_\rho|+|(\Lambda\triangle\varepsilon,\triangle\varepsilon)_\rho|+|(\nabla\triangle\psi,\triangle\varepsilon)_\rho|+|(\nabla\triangle\varepsilon,\triangle\varepsilon)_\rho|\right) \\
\lesssim\ & e^{-2\mu s}\left(\sum_{j=2}^{n+1}|\triangle a_j|^2+\|\triangle\varepsilon\|_{H^1_\rho}^2\right).
\end{aligned}
$$

Finally, the pointwise estimate (4.64) and Cauchy-Schwarz imply for the nonlinear term

$$
\left|\left((\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v,\triangle\varepsilon\right)_\rho\right|\lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}^2+\sum_{j=2}^{n+1}|\triangle a_j|^2\right).
$$

We inject all the above bounds in the identity (4.67), which for s_0 large enough imply the desired estimate (4.66) since $0<c\leq 1$.

step 3 Lipschitz bound by reintegration. We define

$$
A:=\sup_{s\geq s_0}\sum_{j=2}^{n+1}|\triangle a_j(s)|e^{\mu s}<+\infty,\quad \mathcal{E}:=\sup_{s\geq s_0}\|\Delta\varepsilon\|_{L^2_\rho}^2 e^{2\mu s}<+\infty, \tag{4.68}
$$

which are finite from (4.17) and (4.18).

Identity for $\triangle a_j$. Fix j with $2\leq j\leq n+1$. Reintegrating the modulation equation (4.62) yields

$$
\begin{aligned}
\triangle a_j &= \triangle a_j(0)e^{\mu_j(s-s_0)}+e^{\mu_j s}\int_{s_0}^s e^{-\mu_j s'}O(e^{-c\mu s'}(\|\Delta\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|))ds' \\
&= \triangle a_j(0)e^{\mu_j(s-s_0)}+e^{\mu_j s}\int_{s_0}^s O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds' \\
&= \left(\triangle a_j(0)e^{-\mu_j s_0}+\int_{s_0}^{+\infty}O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'\right)e^{\mu_j s} \\
&\quad -e^{\mu_j s}\int_s^{+\infty}O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'.
\end{aligned} \tag{4.69}
$$

The integral appearing in this identity is indeed convergent and satisfies:

$$
\left|\int_s^{+\infty}O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'\right|\lesssim e^{-(\mu_j+(c+1)\mu)s}(A+\sqrt{\mathcal{E}}).
$$

From (4.68) one gets $|\triangle a_j|\lesssim e^{-\mu s}$ and from the two above identities one necessarily must have that the parameter in front of the diverging term $e^{\mu_j s}$ is 0:

$$
\triangle a_j(0)e^{-\mu_j s_0}+\int_{s_0}^{+\infty}O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'=0
$$

which gives the first bound

$$
|\triangle a_j(0)|\lesssim e^{-(c+1)\mu s_0}(A+\sqrt{\mathcal{E}}), \tag{4.70}
$$

and going back to the identity (4.69) one obtains:

$$|\triangle a_j| \lesssim e^{-((c+1)\mu)s}(A+\sqrt{\mathcal{E}})$$

which implies from the definition (4.68) of A the bound

$$A \lesssim e^{-c\mu s_0}\sqrt{\mathcal{E}}. \tag{4.71}$$

Identity for $\triangle\varepsilon$. We reintegrate the energy bound (4.66) to find

$$\begin{array}{lll} \|\triangle\varepsilon\|^2_{L^2_\rho} & \lesssim & \|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{-c_n(s-s_0)}+e^{-c_ns}\int_{s_0}^s e^{c_ns'}\sum_{j=2}^{n+1}|\triangle a_j|^2e^{-\mu cs'}ds' \\ & \lesssim & \|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{-c_n(s-s_0)}+A^2e^{-(c+2)\mu s} \end{array}$$

since $\mu=\frac{c_n}{4}$ from (4.49) and $0<c\ll 1$ can be chosen arbitrarily small. Injecting (4.71) in the above identity yields

$$\mathcal{E}\lesssim \|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{2\mu s_0}$$

so that (4.71) can be rewritten as $A\lesssim \|\triangle\varepsilon(0)\|_{L^2_\rho}e^{(1-c)\mu s_0}$. We inject these two last bounds in (4.70) which finally yields the desired estimate (4.60). □

Appendix A. Coercivity estimates

Lemma A.1 (Weighted L^2 estimate)**.** *Let $u,\partial_r u\in L^2_\rho(\mathbb{R}^3)$, then*

$$\|ru\|_\rho \lesssim \|u\|_{H^1_\rho}. \tag{A.1}$$

Moreover,

$$\|\Delta u\|^2_{L^2_\rho}\lesssim \|-\Delta u+y\cdot\nabla u\|^2_{L^2_\rho}+\|u\|^2_{H^1_\rho}. \tag{A.2}$$

Proof. We may assume by density $u\in\mathcal{D}(\mathbb{R}^3)$.

step 1 Proof of (A.1). We use $\partial_r\rho=-r\rho$ and integrate by parts to compute:

$$\begin{array}{ll} & \int_0^{+\infty}\left(\partial_r u-\frac12 ru\right)^2\rho r^2dr \\ = & \int_0^{+\infty}(\partial_r u)^2\rho r^2dr+\frac14\int_0^{+\infty}r^2u^2\rho r^2dr-\int_0^{+\infty}ru\partial_r u\rho r^2dr \\ = & \int_0^{+\infty}(\partial_r u)^2\rho r^2dr+\frac14\int_0^{+\infty}r^2u^2\rho r^2dr-\frac12\left[r^3\rho u^2\right]_0^{+\infty} \\ & +\frac12\int_0^{+\infty}u^2(3-r^2)\rho r^2dr \\ = & \int_0^{+\infty}(\partial_r u)^2\rho r^2dr-\frac14\int_0^{+\infty}r^2u^2\rho r^2dr+\frac32\int_0^{+\infty}u^2\rho r^2dr \end{array}$$

and hence

$$\|ru\|^2_{L^2_\rho}=\int_0^{+\infty}r^2u^2\rho r^2dr\le 4\int_0^{+\infty}(\partial_r u)^2\rho r^2dr+6\int_0^{+\infty}u^2\rho r^2dr\lesssim\|u\|^2_{H^1_\rho}$$

which concludes the proof of (A.1).

step 2. Proof of (A.2). We compute:

$$\|-\Delta u+y\cdot\nabla u\|^2_{L^2_\rho}=\|\Delta u\|^2_{L^2_\rho}+\|y\cdot\nabla u\|^2_{L^2_\rho}-2\int(\Delta u)y\cdot\nabla u\rho dy.$$

To compute the crossed term, let $u_\lambda(y)=u(\lambda y)$, then

$$\int|\nabla u_\lambda(y)|^2\rho dy=\frac{1}{\lambda}\int|\nabla u(y)|^2\rho\left(\frac{y}{\lambda}\right)dy$$

and hence differentiating in λ and evaluating at $\lambda=1$:

$$2\int\nabla u\cdot\nabla(y\cdot\nabla u)\rho dy=\int|\nabla u|^2(-\rho-y\cdot\nabla\rho)dy$$

i.e.

$$2\int y\cdot\nabla u(\rho\Delta u+\nabla u\cdot\nabla\rho)=\int|\nabla u|^2(\rho+y\cdot\nabla\rho)dy$$

which using $\nabla\rho=-y\rho$ becomes:

$$-2\int(\Delta u)y\cdot\nabla u\rho dy=\int|\nabla u|^2\rho|y|^2-2\int|y\cdot\nabla u|^2\rho-\int\rho|\nabla u|^2.$$

Hence:

$$\begin{aligned}\|-\Delta u+y\cdot\nabla u\|^2_{L^2_\rho} &= \|\Delta u\|^2_{L^2_\rho}+\int\rho(|y|^2|\nabla u|^2-|y\cdot\nabla u|^2)-\int\rho|\nabla u|^2\\ &\geq \|\Delta u\|^2_{L^2_\rho}-\|\nabla u\|^2_{L^2_\rho}\end{aligned}$$

which concludes the proof of (A.2). □

We now turn to the proof of Hardy type inequalities. All proofs are more or less standard and we give the argument for the sake of completeness.

Lemma A.2 (Radial Hardy with best constants)**.** *Let $u\in\mathcal{C}^\infty_c(r>1)$ and*

$$\gamma\neq-1, \tag{A.3}$$

then

$$\int_1^{+\infty}\frac{(\partial_r u)^2}{r^\gamma}dr\geq\left(\frac{\gamma+1}{2}\right)^2\int_1^{+\infty}\frac{u^2}{r^{\gamma+2}}dr. \tag{A.4}$$

Proof. We integrate by parts:

$$\int_1^{+\infty}\frac{u^2}{r^{\gamma+2}}dr=\frac{2}{\gamma+1}\int_1^{+\infty}\frac{u\partial_r u}{r^{\gamma+1}}dr\leq\frac{2}{|\gamma+1|}\left(\int_1^{+\infty}\frac{u^2}{r^{\gamma+2}}dr\right)^{\frac12}\left(\int_1^{+\infty}\frac{(\partial_r u)^2}{r^\gamma}dr\right)^{\frac12}$$

and (A.4) follows. □

Lemma A.3 (Global Hardy for Δ)**.** *Then there exists $c>0$ such that $\forall u\in C^\infty_c(|x|>1)$,*

$$\int|\Delta u|^2dx\geq c\int\left(\frac{|\nabla u|^2}{|x|^2}+\frac{|u|^2}{|x|^4}\right)dx. \tag{A.5}$$

Proof. We decompose u in spherical harmonics and consider

$$\Delta_m u_m=\partial_r^2u_m+\frac{2}{r}\partial_r u_m-\frac{m(m+1)}{r^2},\quad m\in\mathbb{N}.$$

We claim that for all $v\in\mathcal{C}^\infty_c((1,+\infty))$,

$$\int_1^{+\infty}|\Delta_m v|^2r^2dr\geq c\int_1^{+\infty}\left(\frac{|\partial_r v|^2}{r^2}+\frac{(1+m^4)|v|^2}{r^4}\right)r^2dr \tag{A.6}$$

with c independent of m. Assume (A.6), then

$$\int\frac{|\nabla u|^2}{r^2}dx\sim\sum_{m\geq0}\sum_{k=-m}^{m}\int\left(\frac{|\partial_r u_{m,k}|^2}{r^2}+\frac{m^2|u_{m,k}|^2}{r^4}\right)r^2dr$$

and hence summing (A.6) ensures (A.5).
To prove (A.6), we factorize the Laplace operator:

$$\Delta_m = -A_m^*A_m \quad \text{with} \quad \left| \begin{array}{l} A_m = -\partial_r - \frac{\gamma_m}{r} = -\frac{1}{r^{\gamma_m}}\partial_r(r^{\gamma_m}), \quad \gamma_m = -m, \\ A_m^* = \partial_r + \frac{2-\gamma_m}{r}\partial_r = \frac{1}{r^{2-\gamma_m}}\partial_r(r^{2-\gamma_m}). \end{array} \right.$$

Hence from (A.4):

$$\begin{aligned}
&\int_1^{+\infty}(\Delta_m v)^2r^2dr = \int_1^{+\infty}(A_m^*A_m v)^2r^2dr = \int_1^{+\infty}\frac{1}{r^{2-2\gamma_m}}(\partial_r(r^{2-\gamma_m}A_mv))^2dr \\
\geq\ & \left(\frac{2-2\gamma_m+1}{2}\right)^2\int_1^{+\infty}(A_mv)^2dr = \left(\frac{2-2\gamma_m+1}{2}\right)^2\int_1^{+\infty}\frac{1}{r^{2\gamma_m}}(\partial_r(r^{\gamma_m}v))^2dr \\
\geq\ & \left(\frac{2-2\gamma_m+1}{2}\right)^2\left(\frac{2\gamma_m+1}{2}\right)^2\int_1^{+\infty}\frac{v^2}{r^2}dr
\end{aligned}$$

since $\gamma_m = -m$ with $m \in \mathbb{N}$ which ensures that the forbidden value (A.3) is never attained. We conclude that for some universal constant $\delta > 0$ independent of m:

$$\int_1^{+\infty}(\Delta_m v)^2r^2dr \geq \delta(1+m^4)\int_1^{+\infty}\frac{v^2}{r^4}r^2dr.$$

Also, since we have also proved that

$$\int_1^{+\infty}|A_mv|^2dr \lesssim \int_1^{+\infty}(\Delta_m v)^2r^2dr,$$

we infer

$$\begin{aligned}
\int_1^{+\infty}\frac{(\partial_r v)^2}{r^2}r^2dr &\lesssim \int_1^{+\infty}|A_mv|^2dr + \gamma_m^2\int\frac{v^2}{r^4}r^2dr \\
&\lesssim \int_1^{+\infty}(\Delta_m v)^2r^2dr
\end{aligned}$$

and (A.6) follows. □

Appendix B. **Proof of** (4.43)

Let

$$0 < \nu < 1, \quad 1 < p_1, p_2, p_3, p_4 < +\infty, \quad \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Using (4.44), we have

$$\begin{aligned}
\|\nabla^\nu(uv)\|_{L^2} &\sim \|uv\|_{\dot{B}^\nu_{2,2}} \\
&\sim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|uv(\cdot-y)-uv(\cdot)\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|u(\cdot-y)(v(\cdot-y)-v(\cdot))\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\quad+\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|v(\cdot)(u(\cdot-y)-u(\cdot))\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\lesssim \|u\|_{L^{p_4}}\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|v(\cdot-y)-v(\cdot)\|_{L^{p_3}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\quad+\|v\|_{L^{p_2}}\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|u(\cdot-y)-u(\cdot)\|_{L^{p_1}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\lesssim \|u\|_{\dot{B}^\nu_{p_1,2}}\|v\|_{L^{p_2}}+\|u\|_{L^{p_4}}\|v\|_{\dot{B}^\nu_{p_3,2}}
\end{aligned}$$

which concludes the proof of (4.43).

Appendix C. Proof of Lemma 3.2

The existence and uniqueness of $\phi_{n,m}, \nu_m$ satisfying (3.9) and (3.13) is well known. Thus, we focus on their behaviour as $r \to +\infty$.

step 1 Inverting $\mathcal{L}_{m,\infty}$. Let γ_m be the solution to

$$\gamma_m^2-\gamma_m+pc_\infty^{p-1}-m(m+1)=0,$$

the corresponding discriminant Δ_m is given by

$$\Delta_m := 1-4pc_\infty^{p-1}+4m(m+1). \tag{C.1}$$

For $m=1$,

$$\Delta_1=\left(\frac{p+3}{p-1}\right)^2>0 \tag{C.2}$$

and hence for all $m\ge 1$

$$\Delta_m\ge\Delta_1>0.$$

Therefore, γ_m is real and we choose the smallest root[22] so that γ_m is given by

$$\gamma_m=\frac{1-\sqrt{\Delta_m}}{2}.$$

We now solve

$$\mathcal{L}_{\infty,m}(\psi)=0$$

[22] This is motivated by the fact that we obtain below the Kummer's equation with $b=-\gamma_m+1/2$. This is equivalent to $-b=\pm\sqrt{\Delta_m}$. Since the Kummer function is not defined for $-b\in\mathbb{N}$, this justifies to consider the smallest root γ_m.

through the change of variable and unknown

$$\psi(r)=\frac{1}{(2z)^{\frac{\gamma_m}{2}}}w(z),\quad z=\frac{r^2}{2}$$

which leads to

$$\mathcal{L}_{\infty,m}(\psi)\ =\ -\frac{2}{(2z)^{\frac{\gamma}{2}}}\left(zw''(z)+\left(-\gamma_m+\frac{3}{2}-z\right)w'(z)-\left(\frac{1}{p-1}-\frac{\gamma_m}{2}\right)w(z)\right).$$

Thus, $\mathcal{L}_{\infty,m}(\psi)=0$ if and only if

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0$$

where we have used the notations

$$a=\frac{1}{p-1}-\frac{\gamma_m}{2},\quad b=-\gamma_m+\frac{3}{2}.$$

Hence w is a linear combination of two special functions, the Kummer's function $M(a,b,z)$ and the Tricomi function $U(a,b,z)$. These special functions have the following asymptotic behavior at infinity (see for example [47]):

$$M(a,b,z)\sim\frac{\Gamma(b)}{\Gamma(a)}z^{a-b}e^z,\quad U(a,b,z)\sim z^{-a}\text{ as } z\to+\infty.$$

This allows us to infer the asymptotic for w for $z\to 0_+$. Finally, since

$$\psi(r)=\frac{1}{r^{\gamma_m}}w\left(\frac{r^2}{2}\right),$$

we infer from the asymptotic of w the following asymptotic behavior for $\psi_{1,m}$ and $\psi_{2,m}$

$$\psi_{1,m}\sim\frac{1}{r^{\frac{2}{p-1}}}\text{ and }\psi_{2,m}\sim r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\text{ as } r\to+\infty.$$

Consider the Wronskian W which is defined as

$$W\ :=\ \psi'_{1,m}\psi_{2,m}-\psi'_{2,m}\psi_{1,m},$$

then without loss of generality since $W'=\left(r-\frac{2}{r}\right)W$

$$W=\frac{1}{r^2}e^{\frac{r^2}{2}}.$$

We deduce using the variation of constants that the solution w to

$$\mathcal{L}_{\infty,m}(u)=f,$$

is given by

$$u\ =\ \left(a_1+\int_r^{+\infty}f\psi_{2,m}r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_{1,m}+\left(a_2-\int_r^{+\infty}f\psi_{1,m}r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_{2,m}.$$

step 2 Basis of $\mathcal{L}_{m,n}$ near $+\infty$. We now construct a solution to $\mathcal{L}_{n,m}(\varphi)=0$ near $+\infty$ by solving:

$$\mathcal{L}_{\infty,m}(\varphi)\ =\ \mathcal{L}_{n,m}(\varphi)+p(\Phi_n^{p-1}-\Phi_*^{p-1})=p(\Phi_n^{p-1}-\Phi_*^{p-1})\varphi$$

ie

$$\begin{aligned}\varphi\ =\ &\left(a_1+\int_r^{+\infty}p(\Phi_n^{p-1}-\Phi_*^{p-1})\varphi\psi_{2,m}r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_{1,m}\\&+\left(a_2-\int_r^{+\infty}p(\Phi_n^{p-1}-\Phi_*^{p-1})\varphi\psi_{1,m}r'^2e^{-\frac{r'^2}{2}}dr'\right)\psi_{2,m}.\end{aligned}$$

To construct the solution φ_1 with the choice $a_1 = 1$ and $a_2 = 0$ we solve the fixed point equation

$$\varphi_1 = \psi_{1,m} + \widetilde{\varphi}_1, \quad \widetilde{\varphi}_1 = \mathcal{G}\left(\widetilde{\varphi}_1\right) \tag{C.3}$$

where

$$\begin{aligned}\mathcal{G}\left(\widetilde{\varphi}\right)(r) \;&=\; \left(\int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1})\left(\psi_{1,m} + \widetilde{\varphi}\right)(r')\psi_{2,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right)\psi_{1,m}\\ &- \left(\int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1})\left(\psi_{1,m} + \widetilde{\varphi}\right)(r')\psi_{1,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right)\psi_{2,m}.\end{aligned}$$

Recall that we have in view of Corollary 2.6

$$\lim_{n\to+\infty}\sup_{r\geq 1} r^{\frac{2}{p-1}}|\Phi_n(r) - \Phi_*(r)| = 0.$$

Thus, for $n \geq N$ large enough, we infer

$$|\Phi_n(r) - \Phi_*(r)| \leq \frac{1}{r^{\frac{2}{p-1}}} \text{ for } r \geq 1.$$

so that

$$|p(\Phi_n^{p-1} - \Phi_*^{p-1})| \lesssim \frac{1}{r^2}.$$

We infer for $r \geq 1$

$$\begin{aligned}\left|\mathcal{G}\left(\widetilde{\varphi}\right)(r)\right| \;&\lesssim\; \frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}\left(\frac{1}{r'^{\frac{2}{p-1}}} + |\widetilde{\varphi}(r')|\right)dr'\right)\\ &+\; r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}} e^{-\frac{r'^2}{2}}\left(\frac{1}{r'^{\frac{2}{p-1}}} + |\widetilde{\varphi}(r')|\right)dr'\right)\\ &\lesssim\; \frac{1}{r^{2+\frac{2}{p-1}}} + \frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}|\widetilde{\varphi}(r')|dr'\right)\\ &+\; r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}} e^{-\frac{r'^2}{2}}|\widetilde{\varphi}(r')|dr'\right)\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{G}\left(\widetilde{\varphi}_{(1)}\right)(r) - \mathcal{G}\left(\widetilde{\varphi}_{(2)}\right)(r)\right| \;&\lesssim\; \frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}|\widetilde{\varphi}_{(1)}(r') - \widetilde{\varphi}_{(2)}(r')|dr'\right)\\ &+\; r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}} e^{-\frac{r'^2}{2}}|\widetilde{\varphi}_{(1)}(r') - \widetilde{\varphi}_{(2)}(r')|dr'\right)\end{aligned}$$

Thus, for $R \geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r\geq R} r^{1+\frac{2}{p-1}}\left|\widetilde{\varphi}\right|(r).$$

Hence, there exists a unique solution $\widetilde{\varphi}_1$ to (C.3) and

$$\sup_{r\geq R} r^{1+\frac{2}{p-1}}\left|\widetilde{\varphi}_1\right|(r) \lesssim 1.$$

Hence, φ_1 satisfies $\mathcal{L}_{n,m}(\varphi_1) = 0$ and

$$\varphi_1 \sim \frac{1}{r^{\frac{2}{p-1}}}, \text{ as } r \to +\infty.$$

The behaviour of the other solution at infinity is computed using the Wronskian relation

$$W=\varphi_1'\varphi_2-\varphi_2'\varphi_1=-\frac{1}{r^2}e^{\frac{r^2}{2}}$$

and hence

$$\left(\frac{\varphi_2}{\varphi_1}\right)'=-\frac{W}{\varphi_1^2}=\frac{1}{r^2\varphi_1^2}e^{\frac{r^2}{2}}$$

from which

$$\varphi_2(r)=\varphi_1(r)\int_1^r\frac{1}{r'^2\varphi_1^2(r')}e^{\frac{r'^2}{2}}dr'\sim r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\text{ as }r\to+\infty$$

and (3.10) is proved.

step 3 Behaviour of ν_m at $+\infty$. First, consider the solution φ to

$$-\partial_r^2\varphi-\frac{2}{r}\partial_r\varphi+\frac{m(m+1)}{r^2}-\frac{pc_\infty^{p-1}}{r^2}\varphi=f. \tag{C.4}$$

The homogeneous equation admits the basis of solutions

$$\varphi_+=\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}},\quad \varphi_-=\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}$$

and the corresponding Wronskian is given by

$$W(r)=\varphi_+'(r)\varphi_-(r)-\varphi_-'(r)\varphi_+(r)=-\frac{1}{r^2}.$$

Using the variation of constants, the solutions to (C.4) are given by

$$\varphi(r)=\left(a_1-\int_r^{+\infty}f\varphi_-r'^2dr'\right)\varphi_++\left(a_2+\int_r^{+\infty}f\varphi_+r'^2dr'\right)\varphi_-.$$

Now, the equation $H_m(\phi)=0$ can be written as

$$-\partial_r^2\phi-\frac{2}{r}\partial_r\phi+\frac{m(m+1)}{r^2}\phi-\frac{pc_\infty^{p-1}}{r^2}\phi=p\left(Q^{p-1}(r)-\frac{c_\infty^{p-1}}{r^2}\right)\phi(r),$$

i.e. (C.4) with

$$f=p\left(Q^{p-1}(r)-\frac{c_\infty^{p-1}}{r^2}\right)\phi(r).$$

We construct the solution $\phi_{m,1}$ to $H_m(\phi_{m,1})=0$ with the choice $a_1=1$ and $a_2=0$ by solving the fixed point equation

$$\phi_{m,1}=\varphi_++\widetilde{\phi},\quad \widetilde{\phi}=\mathcal{F}\left(\widetilde{\phi}\right) \tag{C.5}$$

where

$$\begin{aligned}\mathcal{F}\left(\widetilde{\phi}\right)(r) \;=\; & -\left(\int_r^{+\infty}p\left(Q^{p-1}(r')-\frac{c_\infty^{p-1}}{r'^2}\right)\left(\varphi_++\widetilde{\phi}\right)(r')\varphi_-r'^2dr'\right)\varphi_+\\ & +\left(\int_r^{+\infty}p\left(Q^{p-1}(r')-\frac{c_\infty^{p-1}}{r'^2}\right)\left(\varphi_++\widetilde{\phi}\right)(r')\varphi_+r'^2dr'\right)\varphi_-.\end{aligned}$$

Recall that

$$Q(r)=\frac{c_\infty}{r^{\frac{2}{p-1}}}+\frac{c_1\sin\left(\omega\log(r)+c_2\right)}{r^{\frac{1}{2}}}+o\left(\frac{1}{r^{\frac{1}{2}}}\right)\text{ as }r\to+\infty$$

so that

$$\left|p\left(Q^{p-1}(r)-\frac{c_\infty^{p-1}}{r^2}\right)\right|\lesssim\frac{1}{r^{1+s_c}}\text{ for }r\geq 1.$$

We infer for $r\geq 1$

$$\begin{aligned}\left|\mathcal{F}(\widetilde{\phi})(r)\right| &\lesssim \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\left(\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\left|\widetilde{\phi}\right|(r')\right)\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}dr'\right)\\ &+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\left(\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\left|\widetilde{\phi}\right|(r')\right)\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}dr'\right)\\ &\lesssim \frac{1}{r^{s_c-1}}\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}\right|(r')dr'\right)\\ &+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}\right|(r')dr'\right)\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{F}(\widetilde{\phi}_1)(r)-\mathcal{F}(\widetilde{\phi}_2)(r)\right| &\lesssim \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}_1-\widetilde{\phi}_2\right|(r')dr'\right)\\ &+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}_1-\widetilde{\phi}_2\right|(r')dr'\right).\end{aligned}$$

Thus, for $R\geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r\geq R} r^{\frac{s_c-1}{2}}r^{\frac{1+\sqrt{\Delta_m}}{2}}|\widetilde{\phi}|(r)$$

and yields a unique solution $\widetilde{\phi}$ to (C.5) with

$$\sup_{r\geq R} r^{\frac{s_c-1}{2}}r^{\frac{1+\sqrt{\Delta_m}}{2}}|\widetilde{\phi}|(r)\leq 1.$$

Hence, $\phi_{m,1}$ satisfies $H_m(\phi_{m,1})=0$ and

$$\phi_{m,1}\sim\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}},\text{ as }r\to+\infty. \tag{C.6}$$

The other independent solution $\phi_{m,2}$ to $H_m(\phi_{m,2})=0$ is computed through the Wronskian relation

$$W := \phi'_{m,1}\phi_{m,2}-\phi'_{m,2}\phi_{m,1}=-\frac{1}{r^2}$$

ie

$$\phi_{m,2}(r)=\phi_{m,1}(r)\int_1^r\frac{1}{r'^2\phi^2_{m,1}(r')}dr'\sim\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\text{ as }r\to+\infty.$$

Since ν_m is a linear combination of $\phi_{m,1}$ and $\phi_{m,2}$, we infer

$$\nu_m(r)\sim\frac{c_{m,+}}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\text{ as }r\to+\infty \tag{C.7}$$

for some constant $c_{m,+}$ and $c_{m,-}$.
case $m=1$: By translation invariance

$$H_1(Q')=0\text{ and }Q'(r)=Q''(0)r(1+O(r^2)) \tag{C.8}$$

Hence, by uniqueness of ν_1, we infer

$$\nu_1(r) = \frac{Q'(r)}{Q''(0)} < 0 \quad \text{on} \quad (0,+\infty)$$

where we used from standard ODE arguments $Q''(0) < 0$ and

$$Q' < 0 \quad \text{on} \quad (0,+\infty). \tag{C.9}$$

case $m = 2$: From (C.8), (C.9) and standard Sturm Liouville oscillation arguments for central potentials [49], the quadratic form $(H_1 u, u)$ is positive on $\dot{H}^1_{\text{rad}}(0,+\infty)$ and hence for $m \geq 2$, $H_m > H_1$ is definite positive, and hence $\nu_m > 0$ on $(0,+\infty)$. Moreover, If $c_{m,-} = 0$ in (C.7), then $\nu_m \in \dot{H}^1_{\text{rad}}$ satisfies $(H_m\nu_m, \nu_m) = 0$ which is a contradiction, hence the leading order behaviour (3.13).

step 4 Completing the basis.
case $m = 2$. Let ϕ_m be the solution to $H_m(\phi_m) = 0$ constructed above with the behaviour (C.6). At the origin, the equation $H_m\psi$ reads

$$A^*_m A_m \psi = V\psi,$$

with

$$A_m v = r^m \partial_r \left(\frac{v}{r^m}\right), \quad A^*_m = \frac{v}{r^{m+1}}\partial_r(r^{m+1}v)$$

and $V \in L^\infty$ and hence all solutions on $(0,\delta)$ with $0 < \delta \ll 1$ are of the form

$$\psi = c_0 r^m + \frac{c_1}{r^{m+1}} + r^m \int_r^\delta \frac{d\tau}{\tau^{2m+1}} \int_0^r \tau^{m+1} V\psi d\tau$$

through an elementary fixed point argument. Hence

$$\phi_m = \frac{c_1 + O(r^2)}{r^{m+1}}. \tag{C.10}$$

Assume by contradiction that $c_1 = 0$. Then, the fixed point above leads to $\phi_m = O(r^m)$. Hence ϕ_m is a zero of H_m in $\dot{H}^1{}_{\text{rad}}$ which is a contradiction. Thus, $c_1 \neq 0$ and together with (C.10), we have obtained (3.14).

case $m = 1$. We let ϕ_1 be given by the Wronskian relation

$$\phi_1 = \nu_1(r) \int_r^1 \frac{d\tau}{\tau^2 \nu_1^2(\tau)} d\tau \sim \left| \begin{array}{ll} \frac{c}{r^2} & \text{as} \quad r \to 0, \quad c \neq 0, \\ \frac{1}{r^{\frac{1-\sqrt{\Delta_1}}{2}}} & \text{as} \quad r \to +\infty, \end{array} \right.$$

which is (3.12).

step 5 Proof of (3.16). Let

$$\kappa_{n,m} := \mu_n^{-m} \varphi_{n,m}(\mu_n r).$$

Then, since $\varphi_{n,m}$ satisfies $\mathcal{L}_{n,m}(\varphi_{n,m}) = 0$, we infer

$$-\partial_r^2 \kappa_{n,m} - \frac{2}{r}\partial_r \kappa_{n,m} + \frac{m(m+1)}{r^2}\kappa_{n,m} - p\left(\mu_n^{\frac{2}{p-1}} \Phi_n(\mu_n r)\right)^{p-1} \kappa_{n,m} = -\mu_n^2 \Lambda \kappa_{n,m}.$$

This yields

$$H_m(\kappa_{n,m}) \quad = \quad f_{n,m} := p\left(\left(\mu_n^{\frac{2}{p-1}} \Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right) \kappa_{n,m} - \mu_n^2 \Lambda \kappa_{n,m}.$$

Since $H_m(\nu_m) = 0$, we infer

$$H_m\left(\kappa_{n,m} - \nu_m\right) \quad = \quad f_{n,m}.$$

We let (ν_m, ϕ_m) be the completed fundamental basis for H_m so that

$$\kappa_{n,m} - \nu_m = \left(a_1 - \int_0^r f_{n,m}\phi_m r'^2 dr'\right)\nu_m + \left(a_2 + \int_0^r f_{n,m}\nu_m r'^2 dr'\right)\phi_m.$$

Since

$$\nu_m(r) = r^m(1+O(r^2)) \text{ and } \varphi_{n,m}(r) = r^m(1+O(r^2)) \text{ as } r \to 0_+,$$

we infer

$$\kappa_{n,m}(r) - \nu(r) = O(r^{m+2})$$

and hence (3.12), (3.14) implies $a_1 = a_2 = 0$ and:

$$\kappa_{n,m} - \nu_m = -\left(\int_0^r f_{n,m}\phi_m r'^2 dr'\right)\nu_m + \left(\int_0^r f_{n,m}\nu_m r'^2 dr'\right)\phi_m.$$

In order to estimate $f_{n,m}$, recall from Corollary 2.6 that we have

$$\sup_{r\le r_0}\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}}Q\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_c-1}$$

This yields

$$\sup_{r\le\frac{r_0}{\mu_n}}\left|p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\right| \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}. \tag{C.11}$$

Also, we rewrite $f_{n,m}$ as

$$\begin{aligned} f_{n,m} &= p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\nu_m - \mu_n^2\Lambda\nu_m \qquad \text{(C.12)} \\ &\quad +p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)(\kappa_{n,m}-\nu_m) - \mu_n^2\Lambda(\kappa_{n,m}-\nu_m). \end{aligned}$$

$0 \le r \le 1$. In view of the asymptotic behavior as $r \to 0_+$ (3.12), (3.14) of the basis of solutions ν_m, ϕ_m, and after integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$, we have for $0 \le r \le 1$ using (C.11) and (C.12):

$$\begin{aligned} |\kappa_{n,m} - \nu_m|(r) &\lesssim \mu_n^2 r^2|\kappa_{n,m} - \nu_m|(r) \\ &+ \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2\right)\left(r^{m+2} + r^m\left(\int_0^r |\kappa_{n,m} - \nu_m| r'^{1-m} dr'\right)\right. \\ &+ \left. r^{-m-1}\left(\int_0^r |\kappa_{n,m} - \nu_m| r'^{m+2} dr'\right)\right). \end{aligned}$$

Using again the asymptotic behavior of ν_m as $r \to 0_+$, we infer for all $m \ge 1$

$$\sup_{0\le r\le 1}\frac{|(\kappa_{n,m} - \nu_m)(r)|}{|\nu_m(r)|} \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{C.13}$$

In particular, this yields

$$\int_0^1 |f_{n,m}| r'^{1-m} dr' + \int_0^1 |f_{n,m}| r'^{m+2} dr' \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{C.14}$$

Next, we consider the region $r \geq 1$. In view of the asymptotic behavior at infinity (3.12), (3.14), (3.11), (3.13), after integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$ and using also (C.14), we have

$$
\begin{aligned}
|\kappa_{n,m} - \nu_m| \quad & \lesssim \quad \mu_n^2 r^2 |\kappa_{n,m} - \nu_m| \\
& + \quad \frac{1}{(1+r)^{\frac{1+\sqrt{\Delta_m}}{2}}} \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' \right) \\
& + \frac{1}{(1+r)^{\frac{1-\sqrt{\Delta_m}}{2}}} \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' \right).
\end{aligned}
$$

After integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$, and in view of the asymptotic behavior of ν_m as $r \to +\infty$ as well as (C.11), we deduce

$$
\begin{aligned}
& |(\kappa_{n,m} - \nu_m)(r)| \\
\lesssim \quad & \frac{1}{(1+r)^{\frac{1+\sqrt{\Delta_m}}{2}}} \left(\int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} |\nu_m| + \mu_n^2 |\Lambda \nu_m| + \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) |\kappa_{n,m} - \nu_m| \right) \right. \\
& \left. \times \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \\
& + \frac{1}{(1+r)^{\frac{1-\sqrt{\Delta_m}}{2}}} \left(\int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} |\nu_m| + \mu_n^2 |\Lambda \nu_m| + \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) |\kappa_{n,m} - \nu_m| \right) \right. \\
& \left. \times \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right).
\end{aligned}
$$

case $m \geq 2$: We estimate from (3.13):

$$
\begin{aligned}
& \frac{|(\kappa_{n,m} - \nu_m)(r)|}{|\nu_m(r)|} \\
\lesssim \quad & \frac{1}{(1+r)^{\sqrt{\Delta_m}}} \left\{ \int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \left(\frac{1}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} + |\kappa_{n,m} - \nu_m| \right) \right. \\
& \left. \times \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right\} \\
& + \int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \left(\frac{1}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} + |\kappa_{n,m} - \nu_m| \right) \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' \\
& + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2.
\end{aligned}
$$

This yields

$$
\sup_{1 \leq r \leq \frac{r_0}{\mu_n}} \frac{|(\kappa_{n,m} - \nu_m)(r)|}{|\nu_m(r)|} \quad \lesssim \quad r_0^2 \left(1 + \frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}} \right)
$$

which together with (C.13) concludes the proof of (3.16) for $n \geq N$ large enough and $m \geq 2$.
case $m = 1$ We estimate using (3.11), (3.12):

$$
\begin{aligned}
&\frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|}\\
\lesssim\ & \int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\frac{1}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}+|\kappa_{n,1}-\nu_1|\right)\frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_1}}{2}}}dr'\\
&+\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\\
&+(1+r)^{\sqrt{\Delta_1}}\left(\int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\frac{1}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}+|\kappa_{n,1}-\nu_1|\right)\frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}dr'\right.\\
&\left.+\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right).
\end{aligned}
$$

This yields[23]

$$
\sup_{1\leq r\leq \frac{r_0}{\mu_n}}\frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|} \lesssim r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right)+\left(\frac{r_0}{\mu_n}\right)^{\sqrt{\Delta_1}}\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)
$$

and hence, together with (C.13) and the fact that[24] $\sqrt{\Delta_1} < 2$, we have for $n \geq N$ large enough

$$
\sup_{0\leq r\leq \frac{r_0}{\mu_n}}\frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|} \lesssim r_0^2.
$$

The corresponding estimates for first order derivatives are obtained in the same way, and (3.16) is proved.

Appendix D. Proof of Lemma 3.3

step 1 Proof of (3.18). Let

$$\kappa_n := \varphi_{n,0}(\mu_n r).$$

Then, since $\varphi_{n,0}$ satisfies $\mathcal{L}_{n,0}(\varphi_{n,0}) = 0$, we infer

$$
-\partial_r^2\kappa_n - \frac{2}{r}\partial_r\kappa_n - p\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1}\kappa_n = -\mu_n^2\Lambda\kappa_n.
$$

[23]Here, we use the fact that

$$\sqrt{\Delta_1}-1 = \frac{4}{p-1} < 1$$

since $p > 5$, so that

$$\int_0^r \frac{r'^2}{(1+r')^{1+\sqrt{\Delta_1}}} \lesssim (1+r)^{2-\sqrt{\Delta_1}}.$$

[24]Indeed, we have in view of (C.2)

$$\sqrt{\Delta_1} = \frac{p+3}{p-1} = 2 - \frac{p-5}{p-1} < 2$$

since $p > 5$.

This yields

$$H(\kappa_n) \quad = \quad f_n$$

where we have introduced the notation

$$f_n \quad := \quad p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\kappa_n - \mu_n^2\Lambda\kappa_n.$$

Since $H(\Lambda Q) = 0$, we infer

$$H\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right) \quad = \quad f_n.$$

Recall the solution ρ to $H(\rho) = 0$ constructed in Lemma 2.3 such that $(\Lambda Q, \rho)$ forms a basis of solutions of $H(w) = 0$, then the solution to

$$H(w) = f$$

is given by

$$w \quad = \quad \left(a_1 + \int_0^r f\rho r'^2 dr'\right)\Lambda Q + \left(a_2 - \int_0^r f\Lambda Q r'^2 dr'\right)\rho.$$

We infer

$$\kappa_n - \frac{p-1}{2}\Lambda Q \quad = \quad \left(a_1 + \int_0^r f_n\rho r'^2 dr'\right)\Lambda Q + \left(a_2 - \int_0^r f_n\Lambda Q r'^2 dr'\right)\rho.$$

Since ΛQ is a smooth function at $r = 0$ with

$$\Lambda Q(0) = \frac{2}{p-1} \neq 0,$$

we infer from the Wronskian relation that ρ has the following asymptotic behavior

$$\rho \sim \frac{c}{r} \text{ as } r \to 0_+$$

for some constant $c \neq 0$, and hence, we must have $a_2 = 0$. Furthermore, since we have

$$\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right)(0) = 0, \quad \Lambda Q(0) = \frac{2}{p-1} \neq 0$$

we infer $a_1 = 0$. Hence, we have

$$\kappa_n - \frac{p-1}{2}\Lambda Q \quad = \quad \left(\int_0^r f_n\rho r'^2 dr'\right)\Lambda Q - \left(\int_0^r f_n\Lambda Q r'^2 dr'\right)\rho.$$

In order to estimate f_n, recall from Corollary 2.6 that we have

$$\sup_{r\leq r_0}\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}}Q\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_c-1}$$

This yields

$$\sup_{r\leq\frac{r_0}{\mu_n}}\left|p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\right| \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}. \tag{D.1}$$

Also, we rewrite f_n as

$$\begin{aligned} f_n &= p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\frac{p-1}{2}\Lambda Q - \mu_n^2\frac{p-1}{2}\Lambda^2 Q \\ &\quad + p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right) - \mu_n^2\Lambda\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right). \end{aligned} \tag{D.2}$$

We start with the region $0 \le r \le 1$. In view of the asymptotic behavior for ΛQ and ρ:

$$\Lambda Q \sim \frac{2}{p-1} \text{ and } \rho \sim \frac{c}{r} \text{ as } r \to 0+,$$

we infer

$$\left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| \lesssim \int_0^r |f_n| r' dr' + \frac{1}{r}\left(\int_0^r |f_n| r'^2 dr'\right).$$

Together with (D.1) and (D.2) and integrating by parts the term $\Lambda(\kappa_n - (p-1)/2\Lambda Q)$, we deduce

$$\begin{aligned} \left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| &\lesssim \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2\right)\left(1 + \int_0^r \left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| r' dr'\right. \\ &\quad \left. + \frac{1}{r}\left(\int_0^r \left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| r'^2 dr'\right)\right). \end{aligned}$$

We infer

$$\sup_{0 \le r \le 1}\left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{D.3}$$

In particular, this yields

$$\int_0^1 |f_n| r' dr' + \int_0^1 |f_n| r'^2 dr' \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{D.4}$$

Next, we consider the region $r \ge 1$. Recall the asymptotic behavior at infinity of ΛQ and ρ given by Lemma 2.3

$$\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}}, \quad \rho(r) \sim \frac{c_9 \sin(\omega \log(r) + c_{10})}{r^{\frac{1}{2}}} \text{ as } r \to +\infty,$$

where $c_7, c_9 \neq 0$, $c_8, c_{10} \in \mathbb{R}$. We infer for $r \ge 1$

$$\left|\kappa_n - \frac{p-1}{2}\Lambda Q\right| \lesssim \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 + \int_1^r |f_n| \frac{r'^2}{(1+r')^{\frac{1}{2}}} dr'\right)\frac{1}{(1+r)^{\frac{1}{2}}}.$$

After integrating by parts the term $\Lambda(\kappa_n-(p-1)/2\Lambda Q)$, and together with (D.1) and (D.2), we deduce

$$\begin{aligned}
&(1+r)^{\frac12}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\\
\lesssim\ &\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(1+\int_1^r\left(\frac{1}{(1+r')^{\frac12}}+\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\right)\frac{r'^2}{(1+r')^{\frac12}}dr'\right)\\
\lesssim\ &\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)(1+r)^2+\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\int_1^r\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\frac{r'^2}{(1+r')^{\frac12}}dr'\right).
\end{aligned}$$

This yields

$$\sup_{1\leq r\leq\frac{r_0}{\mu_n}}(1+r)^{\frac12}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\ \lesssim\ r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right)$$

which together with (D.3) implies

$$\sup_{0\leq r\leq\frac{r_0}{\mu_n}}(1+r)^{\frac12}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\ \lesssim\ r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right).$$

Hence, we have for $n\geq N$ large enough

$$\sup_{0\leq r\leq r_0}\left(1+\frac{r}{\mu_n}\right)^{\frac12}\left|\varphi_{n,0}(r)-\frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right|\ \lesssim\ r_0^2.$$

step 2 Proof of (3.19). Recall from Lemma 3.3 that we have for $n\geq N$ large enough

$$\sup_{0\leq r\leq r_0}\left(1+\frac{r}{\mu_n}\right)^{\frac12}\left|\varphi_{n,0}(r)-\frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right|\ \lesssim\ r_0^2.$$

Also, recall that

$$\Lambda Q(r)\sim\frac{c_7\sin(\omega\log(r)+c_8)}{r^{\frac12}}\text{ as }r\to+\infty$$

and that $r_{\Lambda Q,n}<r_0/\mu_n$ introduced in Corollary 2.6 denotes the last zero of ΛQ before r_0/μ_n. This yields

$$\left|\omega\log(r_{1,n})-\omega\log(\mu_n)+c_8-(\omega\log(r_{\Lambda Q,n})+c_8)\right|\lesssim r_0^2$$

and hence

$$r_{1,n}\ =\ \mu_n r_{\Lambda Q,n}e^{O(r_0^2)}=\mu_n r_{\Lambda Q,n}(1+O(r_0^2)).$$

Furthermore, since we have from the proof of Corollary 2.6 that

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n}\leq r_{\Lambda Q,n}\leq\frac{r_0}{\mu_n},$$

and

$$r_{0,n}\ =\ \mu_n r_{\Lambda Q,n}(1+O(r_0^2)),$$

we deduce

$$r_{1,n}=r_{0,n}+O(r_0^3)$$

and

$$e^{-\frac{2\pi}{\omega}}r_0\leq r_{1,n}\leq r_0.$$

References

[1] Budd, C.; Norbury, J., Semilinear elliptic equations and supercritical growth, J. Differential Equations 68 (1987), no. 2, 169–197.

[2] Budd, C.; Qi, Y-W., The existence of bounded solutions of a semilinear elliptic equations, J. Differential Equations 82 (1989) 207–218.

[3] P. Biernat, P. Bizoń, Shrinkers, expanders, and the unique continuation beyond generic blowup in the heat flow for harmonic maps between spheres, Nonlinearity **24** (8), 2011, 2211–2228.

[4] Cazenave, T.; Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10, NYU, CIMS, AMS 2003.

[5] Cazenave, T,; Shatah, J.; Tahvildar-Zadeh, A. S., Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields, Ann. Inst. H. Poincaré Phys. Théor. 68 (1998), no. 3, 315–349.

[6] Collot, C, Type II blow up manifolds for the energy supercritical wave equation. arXiv preprint arXiv:1407.4525.

[7] Collot, C, Non radial type II blow up for the energy supercritical semilinear heat equation, preprint 2016.

[8] Collot, C.; Merle. F; Raphaël, P., Dynamics near the ground state for the energy critical nonlinear heat equation in large dimension, preprint 2016.

[9] Corlette, K., Wald, R. M. Morse Theory and Infinite Families of Harmonic Maps Between Spheres. Communications in Mathematical Physics (2001), 215(3), 591-608.

[10] Côte, R.; Martel, Y.; Merle, F., Construction of multi-soliton solutions for the L2-supercritical gKdV and NLS equations, Rev. Mat. Iberoam. 27 (2011), no. 1, 273–302.

[11] Dancer, E. N.; Guo, Z.; Wei, J, Non-radial singular solutions of the Lane-Emden equation in $\mathbb{R}^N$, Indiana Univ. Math. J. 61 (2012), no. 5, 1971–1996.

[12] Ding, W. Y., Ni, W. M. (1985). On the elliptic equation $Deltau + Ku^{(n+2)/(n-2)} = 0$ and related topics. Duke mathematical journal, 52(2), 485-506.

[13] Donninger, R., On stable self-similar blowup for equivariant wave maps, Comm. Pure Appl. Math. 64 (2011), 1095–1147.

[14] Donninger, R., Stable self-similar blowup in energy supercritical Yang-Mills theory, Math. Z. 278 (2014), 1005–1032.

[15] Donninger, R.; Schörkhuber, B.,Stable blow up dynamics for energy supercritical wave equations, Trans. Amer. Math. Soc. 366 (2014), no. 4, 2167–2189.

[16] Donninger, R.; Schörkhuber, B.,Stable blow up for the supercritical Yang-Mills heat flow, arXiv preprint arXiv:1604.07737.

[17] Donninger, R.; Krieger, J.; Szeftel, J.; Wong, W., Codimension one stability of the catenoid under the vanishing mean curvature flow in Minkowski space. Duke Math. J. 165 (2016), 723–791.

[18] Filippas, S., Herrero, M. A., Velazquez, J. J. (2000, December). Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity. In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences (Vol. 456, No. 2004, pp. 2957-2982). The Royal Society.

[19] Germain, P.; Rupflin, M., Selfsimilar expanders of the harmonic map flow, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011), no. 5, 743–773.

[20] Giga, Y. (1986). On elliptic equations related to self-similar solutions for nonlinear heat equations. Hiroshima mathematical journal, 16(3), 539-552.

[21] Giga, Y., Kohn, R. V. (1985). Asymptotically self similar blow up of semilinear heat equations. Communications on Pure and Applied Mathematics, 38(3), 297–319.

[22] Y. Giga and R.V. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J., 36 (1987), 1–40.

[23] Giga, Y., Kohn, R. V. (1989). Nondegeneracy of blowup for semilinear heat equations. Communications on Pure and Applied Mathematics, 42(6), 845–884.

[24] Giga, Y., Matsui, S. Y., Sasayama, S. (2002). Blow up rate for semilinear heat equation with subcritical nonlinearity.

[25] M. Herrero, J. Velazquez, A blow up result for the semilinear heat equations in the supercritical case, preprint 1992.

[26] Hadzic, M.; Raphaël, P., On melting and freezing for the radial Stefan problem, preprint 2015.

[27] Joseph, D. D., Lundgren, T. S. (1973). Quasilinear Dirichlet problems driven by positive sources. Archive for Rational Mechanics and Analysis, 49(4), 241-269.

[28] Krieger, J.; Schlag, W., Non-generic blow-up solutions for the critical focusing NLS in 1-D. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 1, 1–125.
[29] H. Koch, Self-similar solutions to super-critical gKdV, Nonlinearity **28** (3), 2015, 545–575.
[30] Lepin L.A., Self-similar solutions of a semilinear heat equation, Mat. Model. 2 (1990) 63–7.
[31] Y. Li. Asymptotic Behavior of Positive Solutions of Equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$. J. Differential Equations, 95(2):304-330, 1992.
[32] Martel, Y.; Merle, F., A Liouville theorem for the critical generalized Korteweg-de Vries equation, J. Math. Pures Appl. (9) 79 (2000), no. 4, 339–425.
[33] Martel, Y.; Merle, F.; Nakanishi, K.; Raphaël, P., Codimension one threshold manifold for the critical gKdV equation, arXiv:1502.04594.
[34] Martel, Y.; Merle, F.; Raphaël, P., Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton, Acta Math. 212 (2014), no. 1, 59–140.
[35] Matano, H.; Merle, F., Classification of type I and type II behaviors for a supercritical nonlinear heat equation, J. Funct. Anal. 256 (2009), no. 4, 992–1064.
[36] Matano, H.; Merle, F., On nonexistence of type II blowup for a supercritical nonlinear heat equation, Comm. Pure Appl. Math. 57 (2004), no. 11, 1494–1541.
[37] Matano, H., Merle, F., Threshold and generic type I behaviors for a supercritical nonlinear heat equation. Journal of Functional Analysis (2011), 261(3), 716-748.
[38] Merle, F.; Raphaël, P., The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. of Math. (2) 161 (2005), no. 1, 157–222.
[39] Merle, F.; Zaag, H., Stability of the blow-up profile for equations of the type $u_t = \Delta u+|u|^{p-1}u$, Duke Math. J. 86 (1997), no. 1, 143–195.
[40] Merle, F., Zaag, H. (1998). Optimal estimates for blowup rate and behavior for nonlinear heat equations. Communications on pure and applied mathematics, 51(2), 139-196.
[41] Merle, F.; Zaag, H., Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension, Amer. J. Math. 134 (2012), no. 3, 581–648.
[42] Merle, F., Raphaël, P.; Rodnianski, I., Type II blow up for the energy super critical NLS, *to appear in Cambridge Math. Jour.*
[43] Mizoguchi, N., Type-II blowup for a semilinear heat equation, Adv. Differential Equations 9 (2004), no. 11-12, 1279–1316.
[44] Mizoguchi, N., Nonexistence of backward self-similar blowup solutions to a supercritical semilinear heat equation, J. Funct. Anal. 257 (2009), no. 9, 2911–2937.
[45] Mizoguchi, N. (2011). Nonexistence of type II blowup solution for a semilinear heat equation. Journal of Differential Equations, 250(1), 26-32.
[46] Nakanishi, K.; Schlag, W., Invariant manifolds and dispersive Hamiltonian evolution equations, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2011.
[47] F. Olver, D. Lozier, R. Boisvert, C. Clark, NIST Handbook of Mathematical Functions, Cambridge, Cambridge University Press, 2010.
[48] Raphaël, P., Rodnianski, I., Stable blow up dynamics for critical corotational wave maps and the equivariant Yang Mills problems, *Publ. Math. Inst. Hautes Etudes Sci.* 115 (2012), 1–122.
[49] Reed, S.; Simon, B., Methods of modern mathematical physics, I-IV, Functional analysis, Second edition, Academic Press, Inc., New York, 1980.
[50] Schweyer, R. Type II blow-up for the four dimensional energy critical semi linear heat equation. Journal of Functional Analysis (2012), 263(12), 3922-3983.
[51] Troy, W., The existence of bounded solutions of a semilinear heat equation, SIAM J. Math. Anal. 18 (1987), 332–336.

Laboratoire J.A. Dieudonné, Université de Nice-Sophia Antipolis, France
E-mail address: `ccollot@unice.fr`

Laboratoire J.A. Dieudonné, Université de Nice-Sophia Antipolis, France
E-mail address: `praphael@unice.fr`

Laboratoire Jacques-Louis Lions, Université Paris 6, France
E-mail address: `jeremie.szeftel@upmc.fr`